

## Lecture 3

# Frequency Domain view of signals

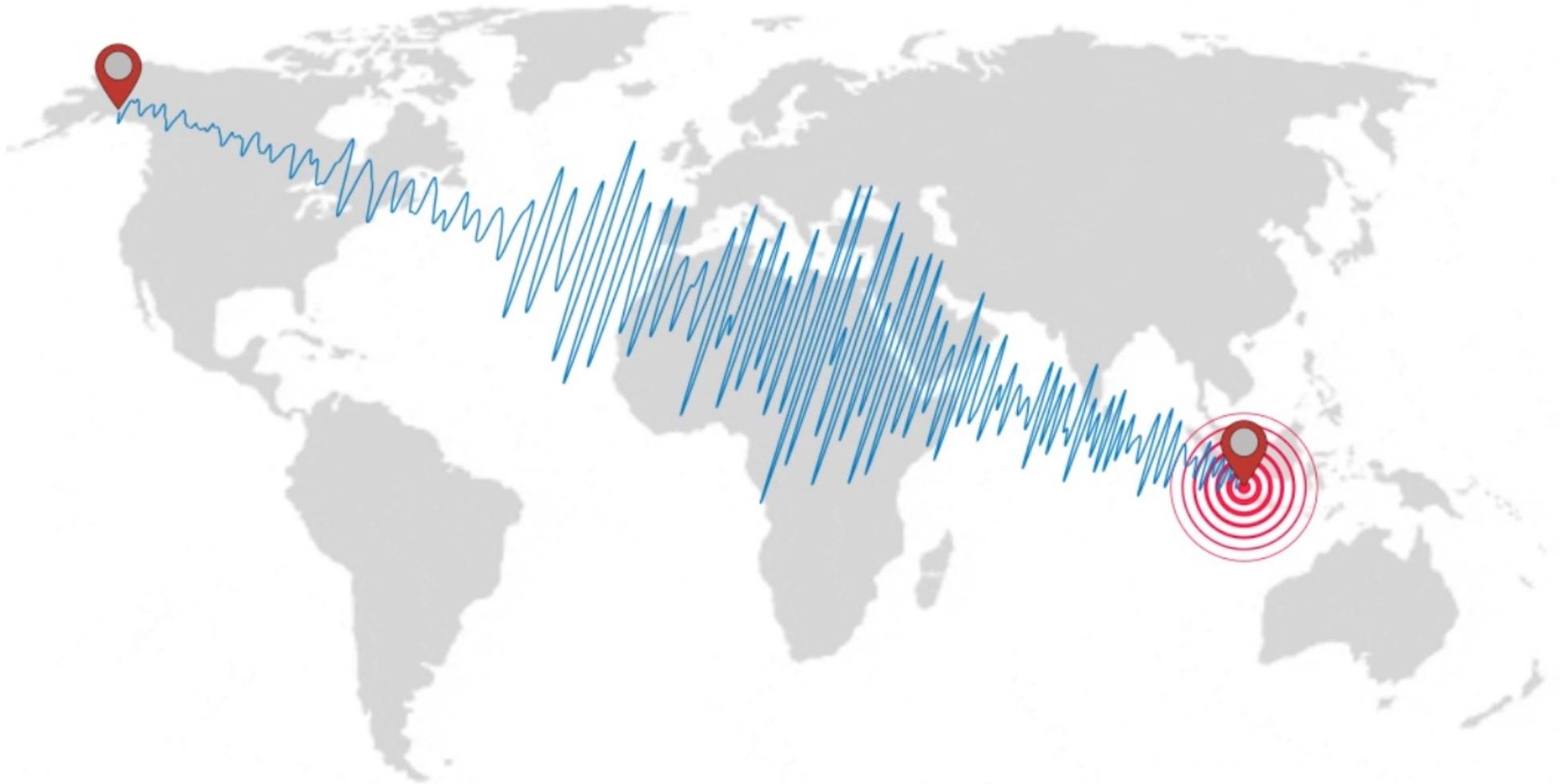
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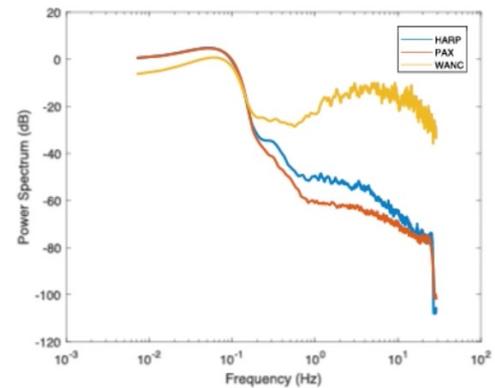
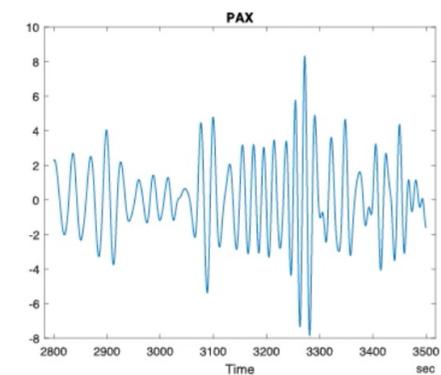
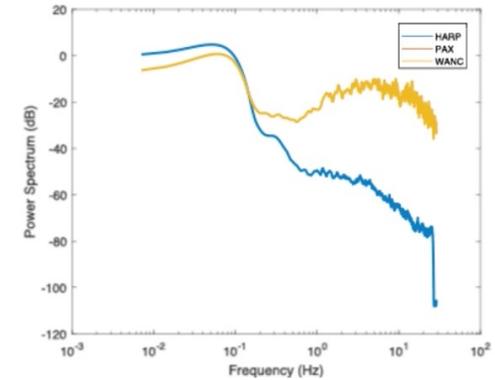
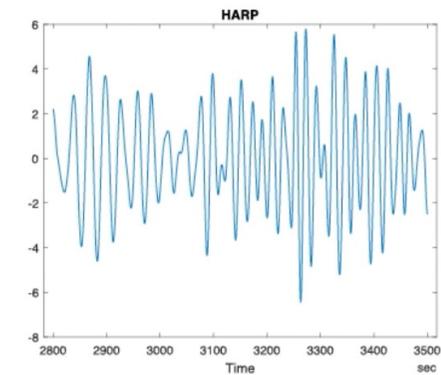
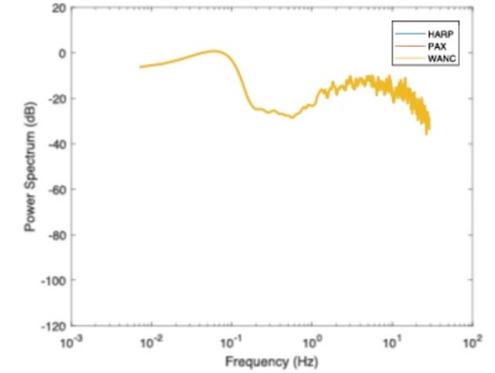
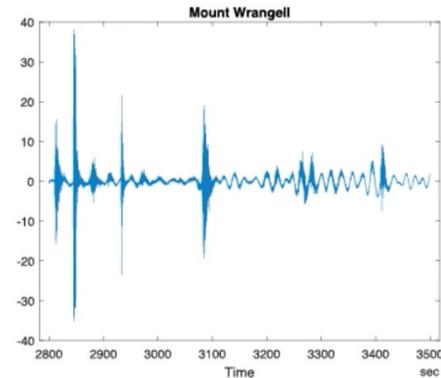
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E-mail: [p.cheung@imperial.ac.uk](mailto:p.cheung@imperial.ac.uk)

## Example – Seismic signal measured in Alaska due to Sumatra Earthquake in 2004

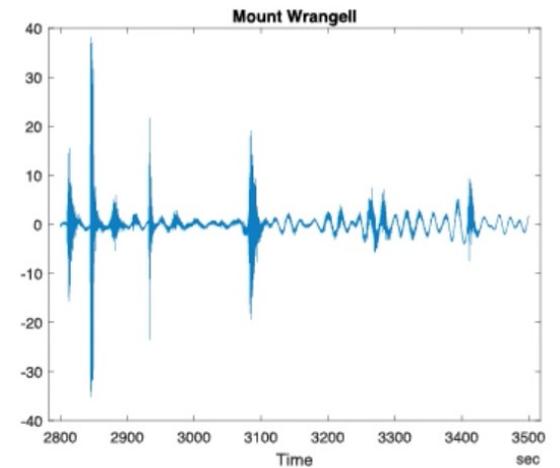
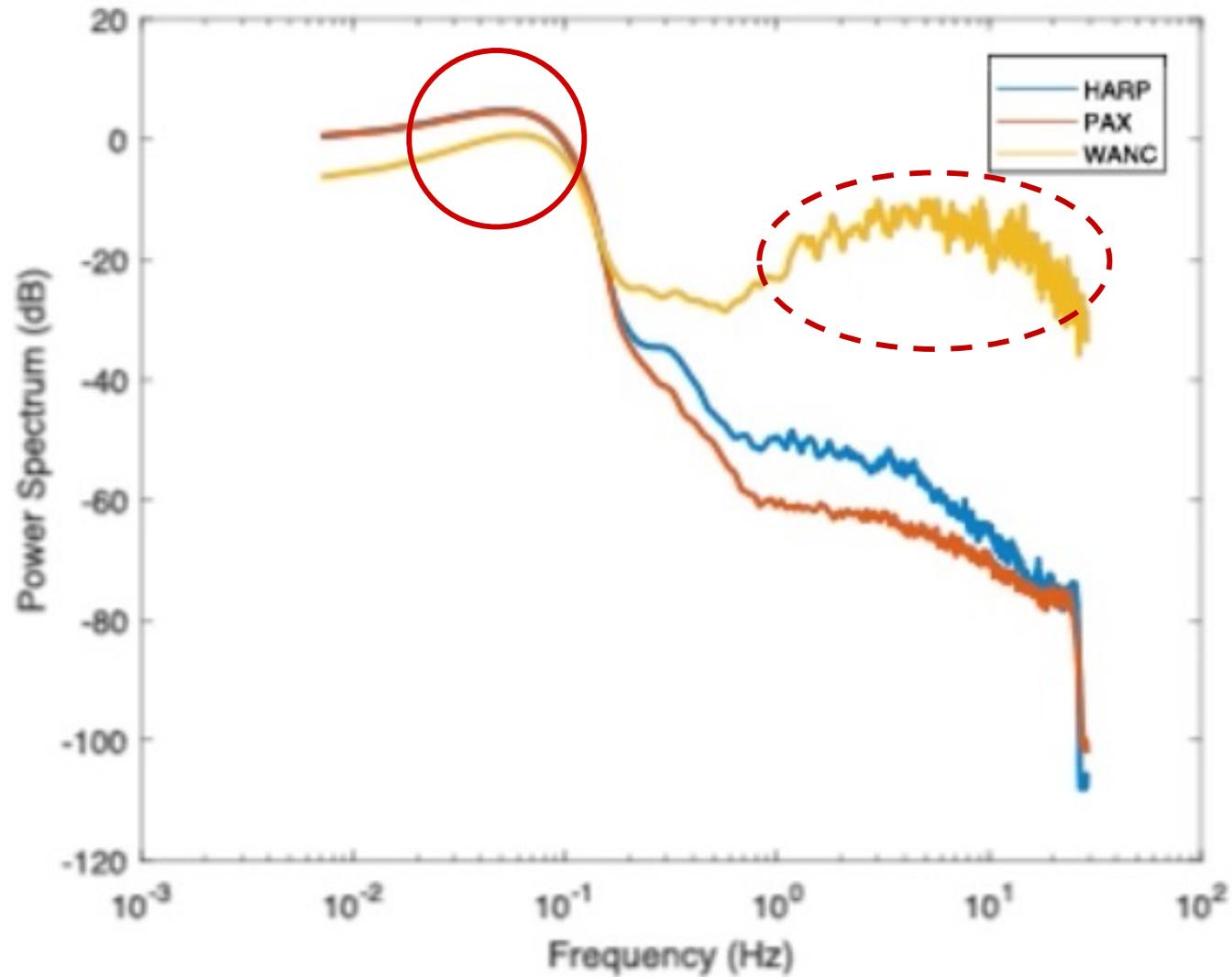
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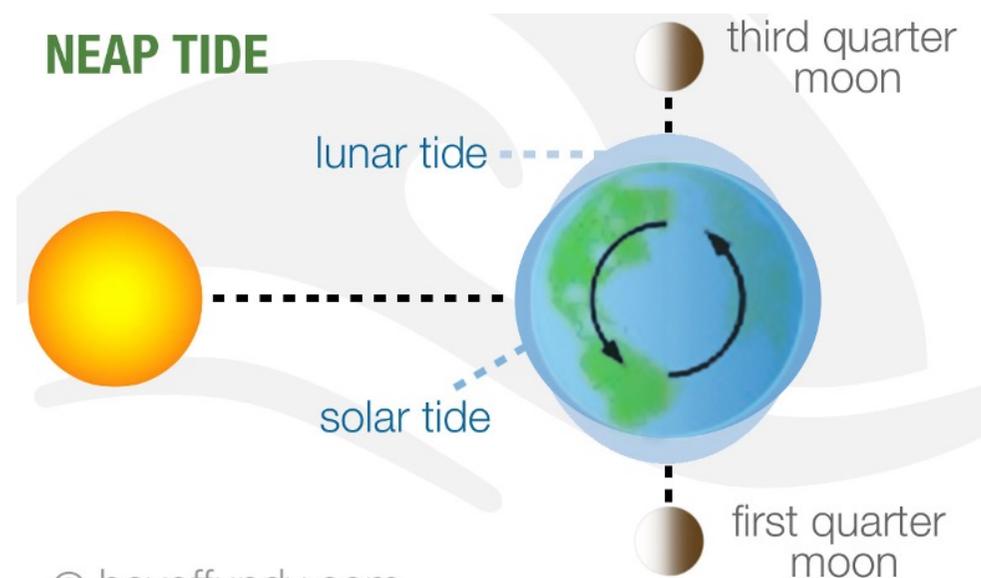
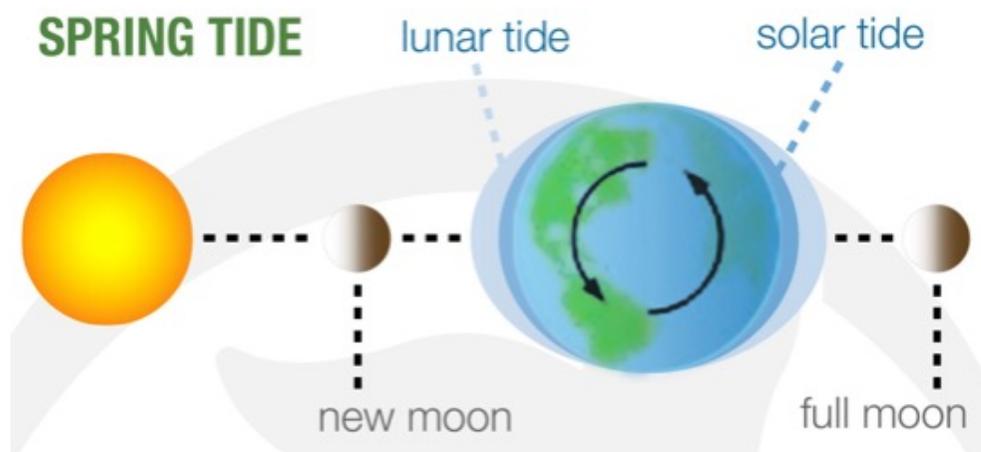
# Three stations recorded data for analysis



# Frequency domain view of signal – more informative



# Prediction of Tides

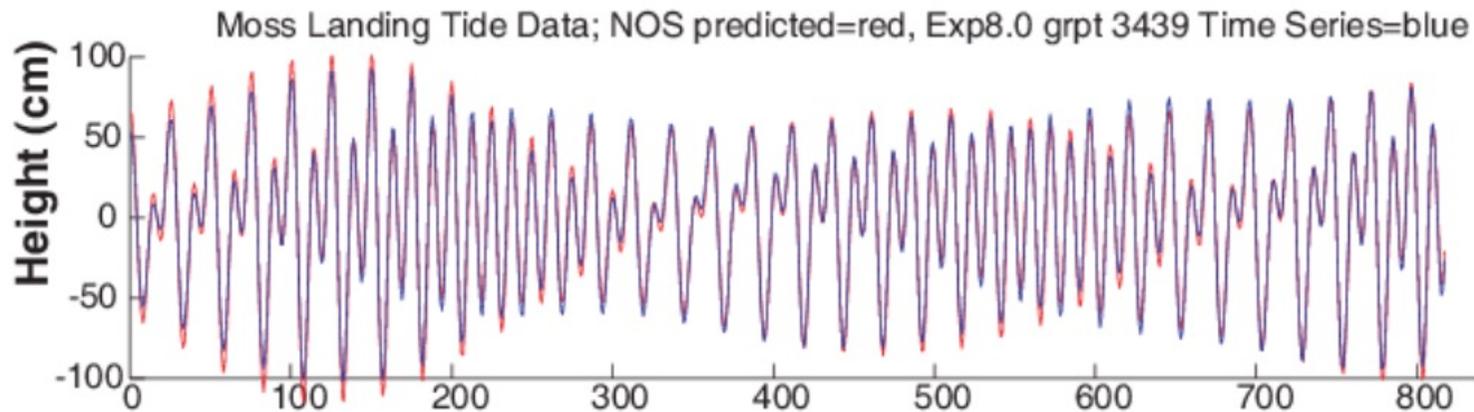
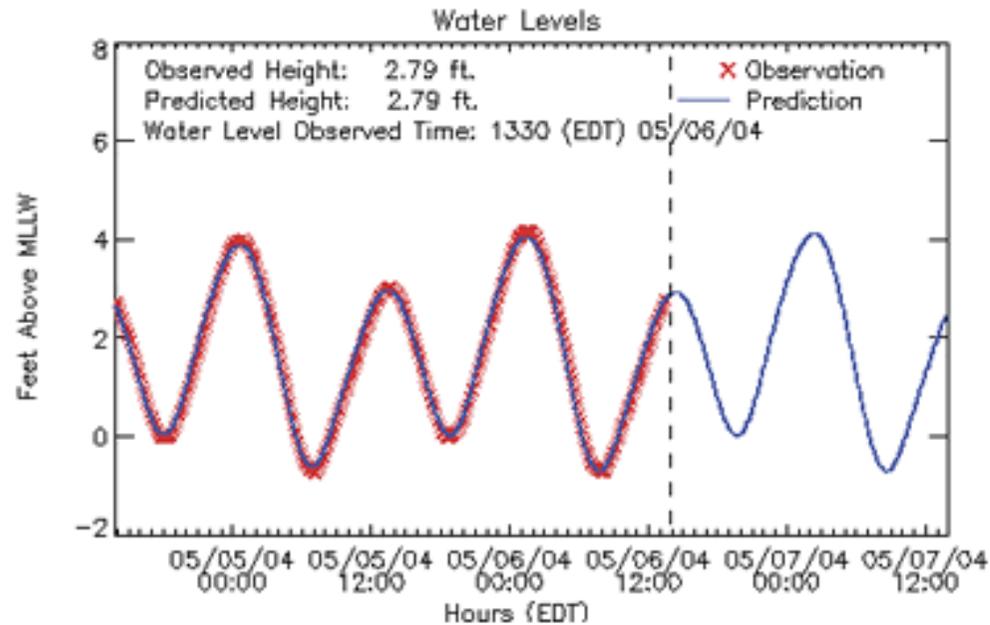


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- ◆ Early scientists:
  - Isaac Newton
  - Joseph Fourier
  - William Thomson (Lord Kelvin)
  - Pierre-Simon Laplace

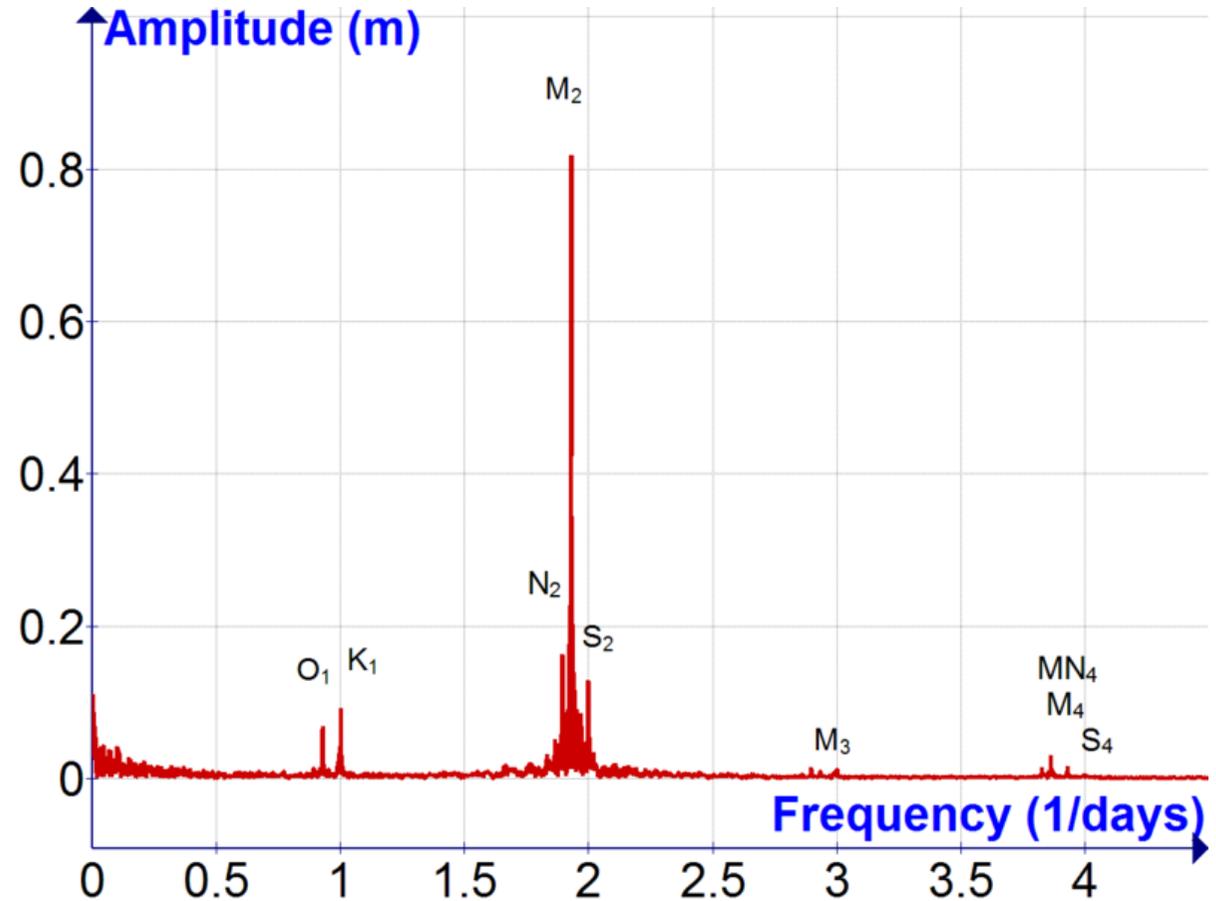
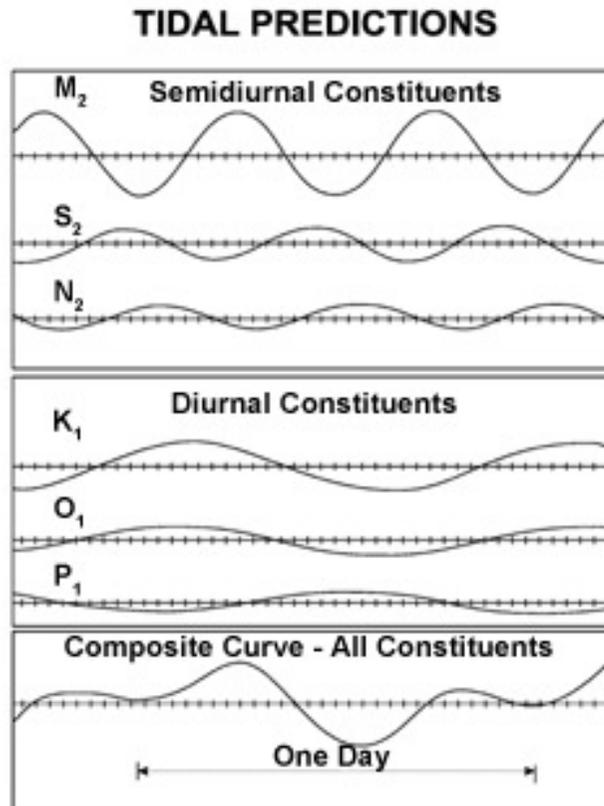
<http://oceanmotion.org/html/background/tides-observing.htm>

# Periodicity of the Tides



<http://oceanmotion.org/html/background/tides-observing.htm>

# Tides decomposed into periodic constituents



# Kelvin's Tide Prediction Machine

First tide prediction computer

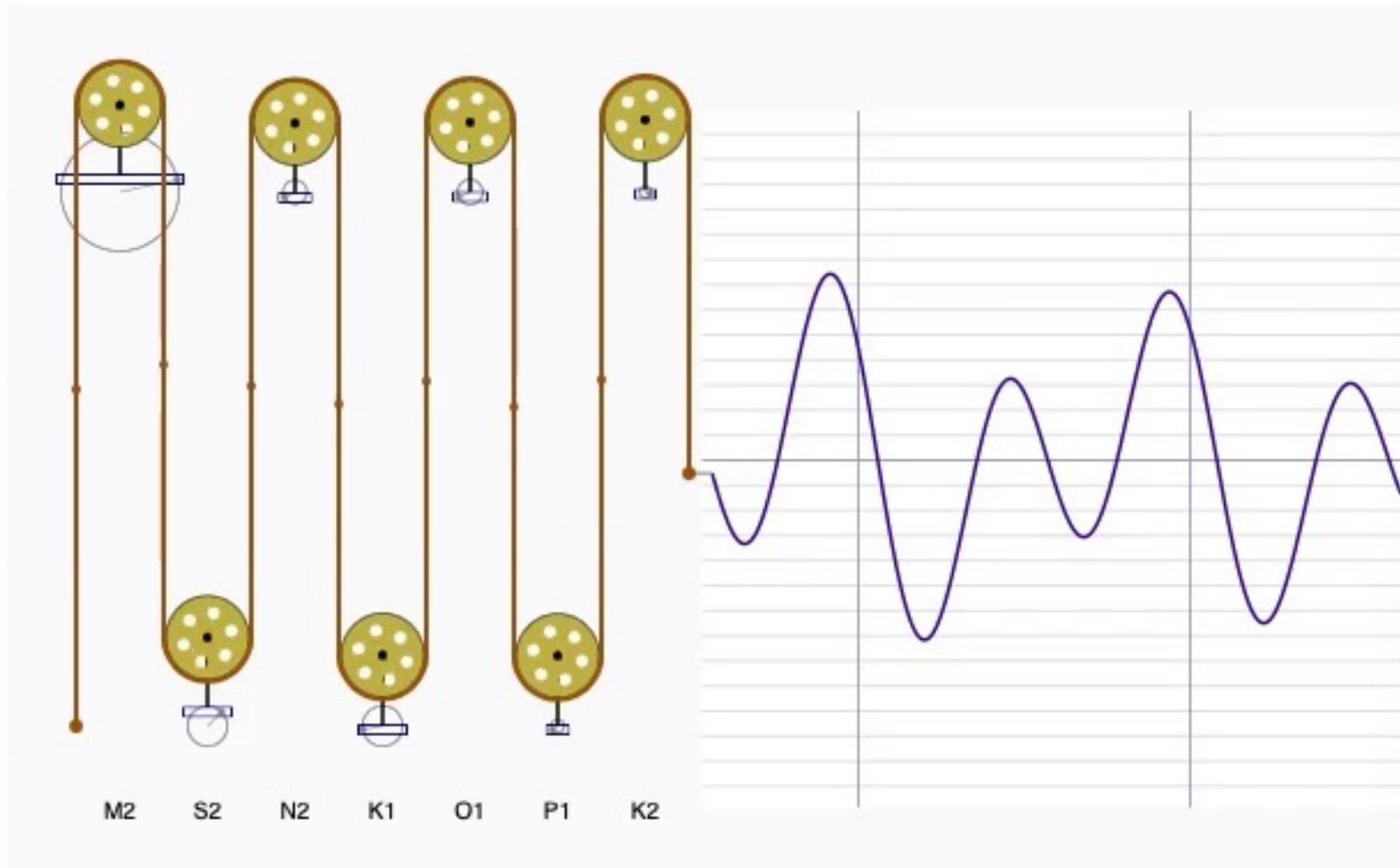


William Thomson (Lord Kelvin)

Doodson\_lege machine in Liverpool



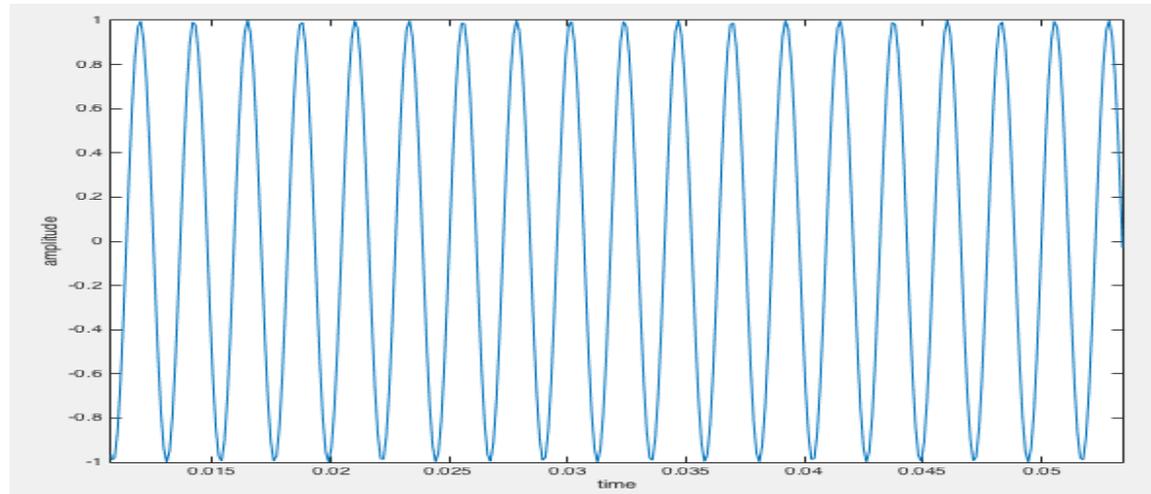
# Animation of the Tide Prediction Machine



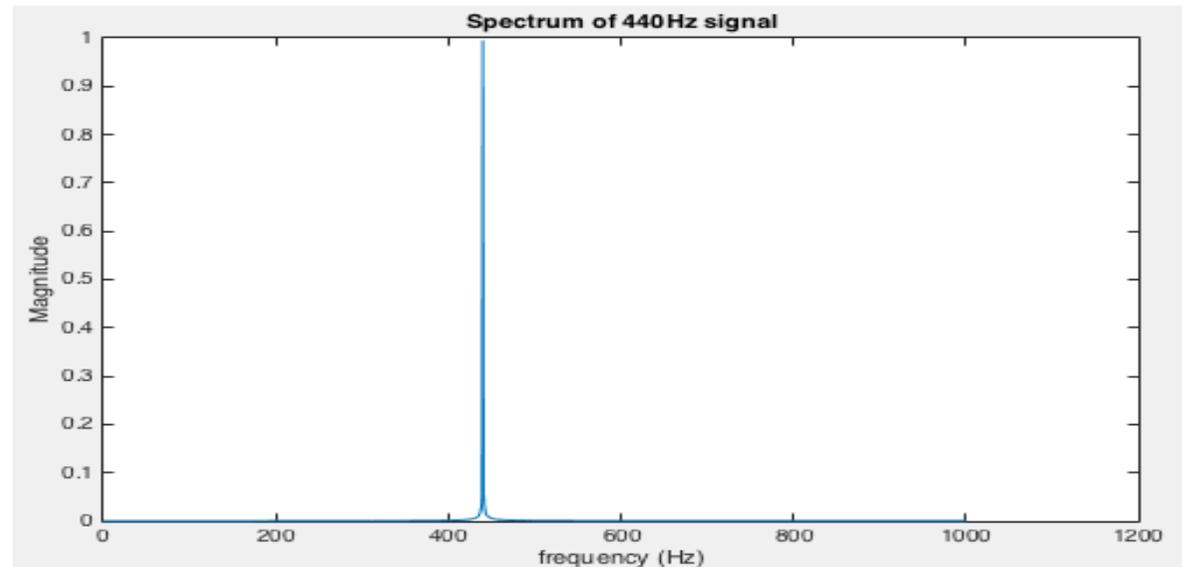
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# Time vs Frequency view of a sinewave

- ◆ Sinewave (sinusoidal signal) in time domain

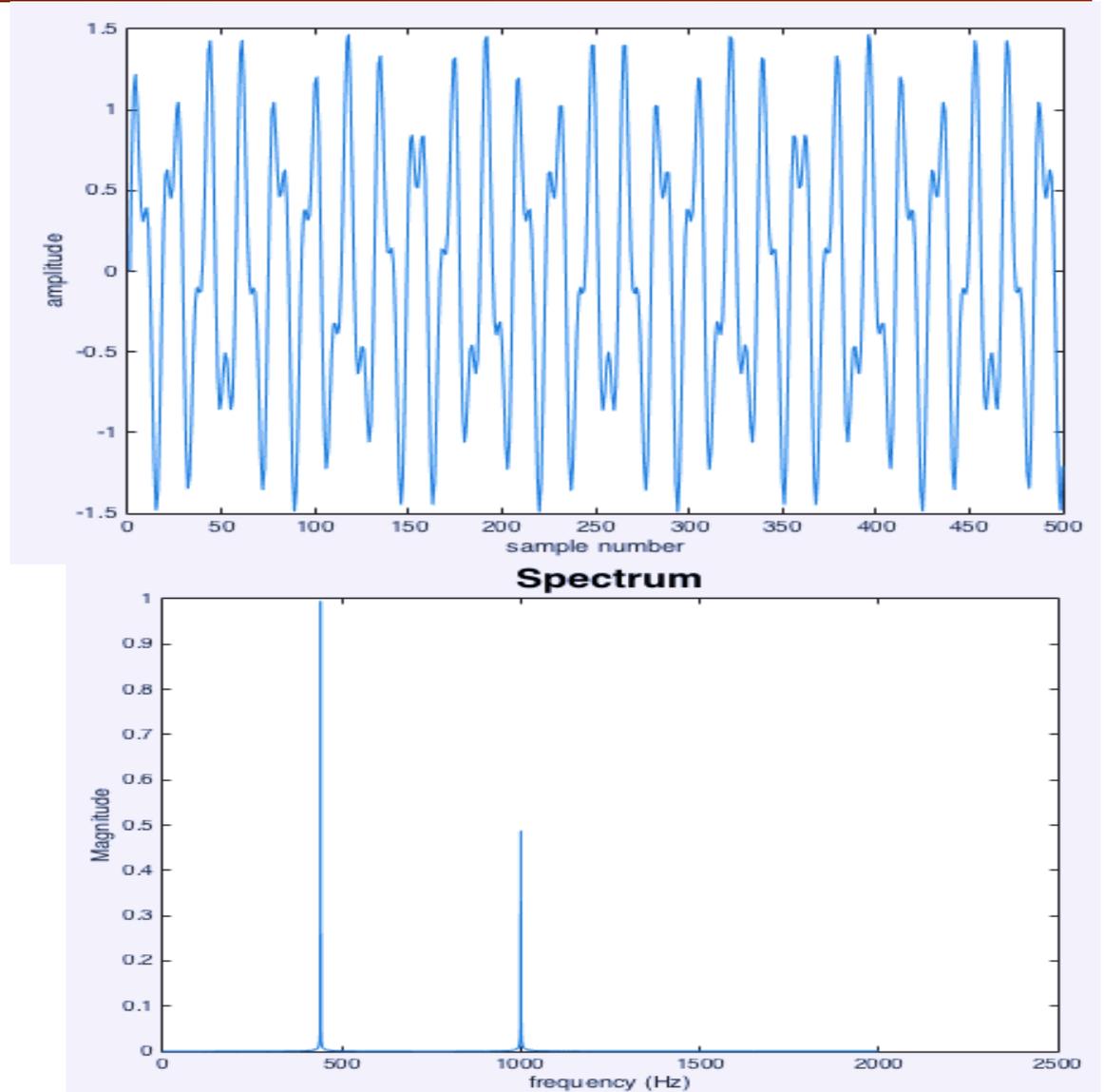


- ◆ Same sinewave in frequency domain



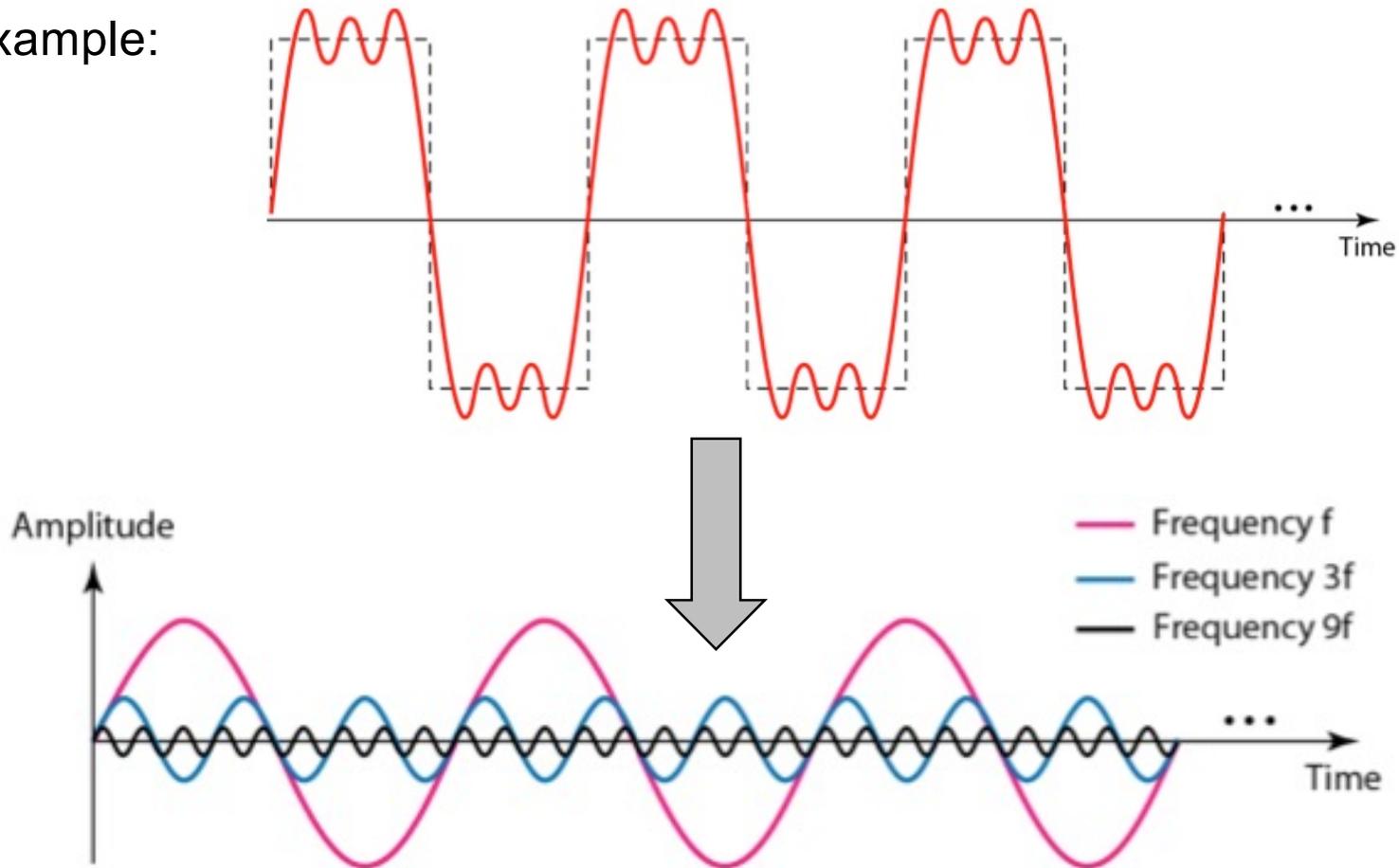
# Two sinewaves

- ◆ Adding 440Hz to 1kHz signal. The 440Hz is twice as large as the 1kHz signal.
- ◆ Spectrum of two sinewaves



# Key idea – Fourier's theory

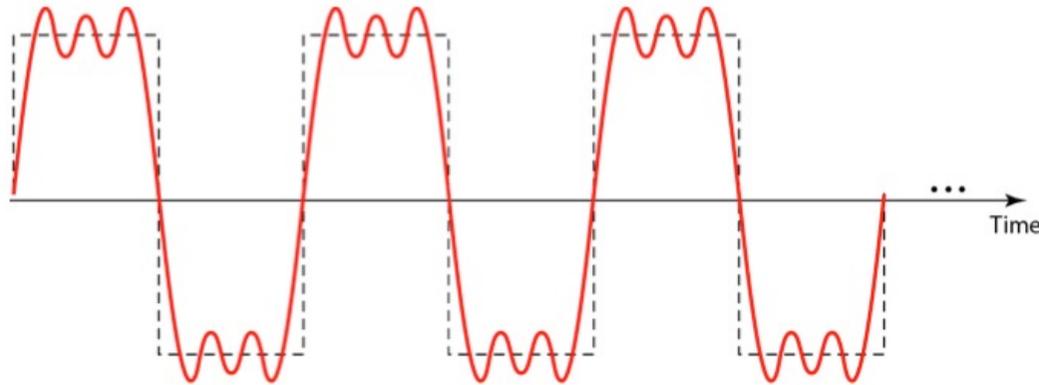
- ◆ Basic idea – any time domain signal can be constructed from **weighted linear sum** of sinusoidal signals (sine or cosine signals) at different frequencies.
- ◆ For example:



# Spectrum – Frequency domain representation

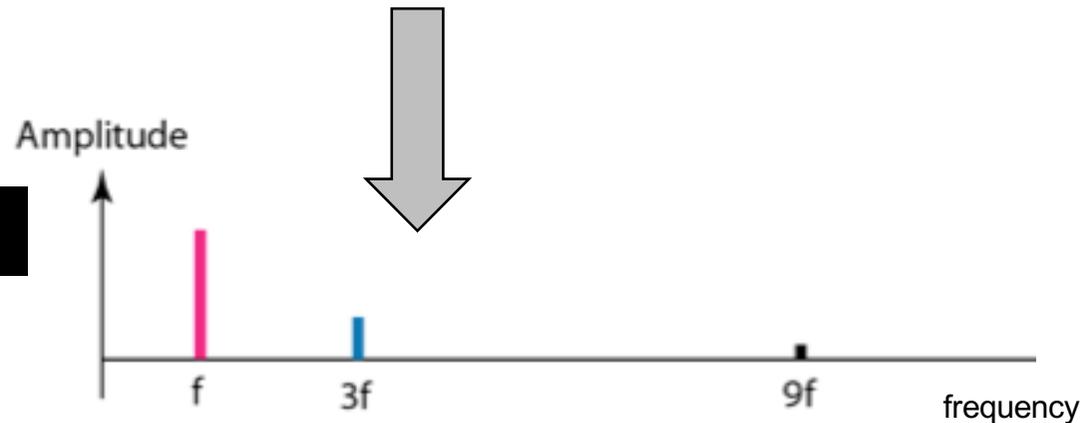
- ◆ Instead of having to store individual time samples, we only need to store the amplitude, frequency and phase of each sinusoidal signal.

Time domain



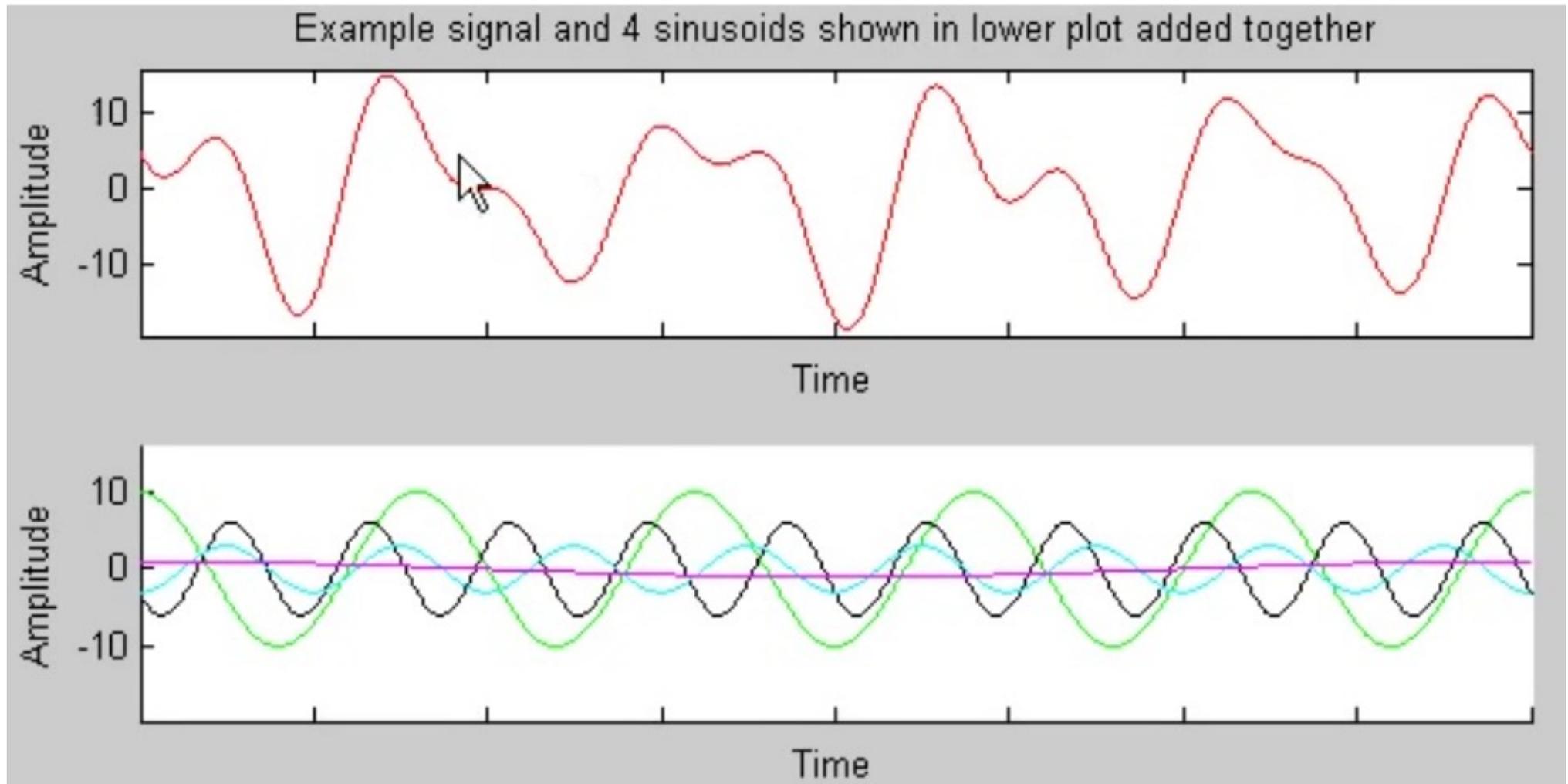
- ◆ Spectrum of signal in frequency domain is represented by amplitude value for each frequency. There is also phase vs frequency, which is not shown here.

Frequency domain



## Another Example

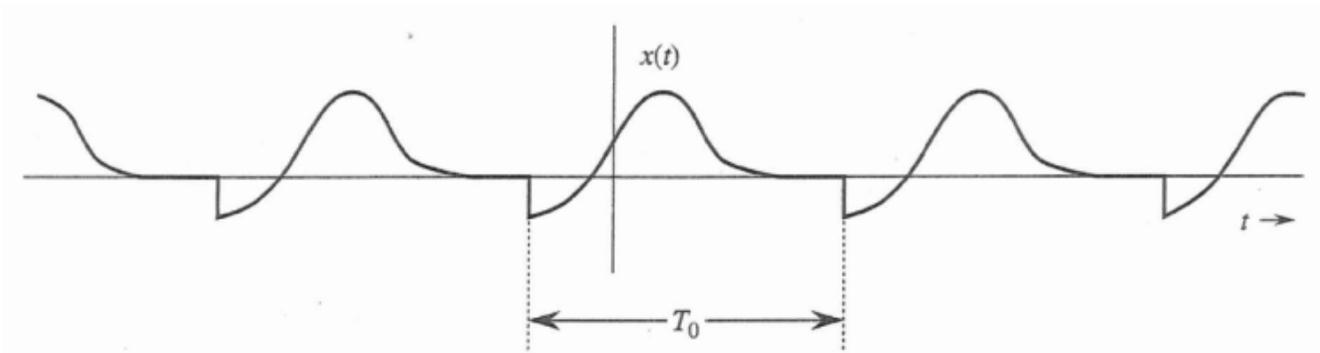
- ◆ Here is another time domain signal that is constructed with four sine waves:



# Periodic Signal & Fourier Series

- ◆ A periodic signal  $x(t)$  with a period of  $T_0$  has the property:

$$x(t) = x(t + T_0) \quad \text{for all } t$$



- ◆ **Fourier series** expresses  $x(t)$  as a **weighted linear sum of sinusoids** (or exponentials) of the fundamental frequency  $f_0 = 1/T_0$  and all its harmonics  $nf_0$  where  $n = 2, 3, 4, \dots$

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{for all integers } n$$

- ◆  $\omega_0$  is called the **fundamental frequency** such that ( $f_0$  in cycles/sec or Hz,  $\omega_0$  in radians/sec)  
 $\omega_0 = 2\pi f_0 = 2\pi / T_0$  and  $n\omega_0$  are the harmonic frequencies
- ◆  $a_0$  is the DC (mean) value of  $x(t)$  and  $a_n, b_n$  are the Fourier coefficients at the frequency  $n\omega_0$

# How to find $a_0$ ?

---

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- ◆ To determine  $a_0$ , we multiply both sides by  $\cos m\omega_0 t$  and integrate over  $T_0$ :

$$\begin{aligned} \int_0^{T_0} x(t) dt &= a_0 \int_0^{T_0} dt \\ &+ \sum_{n=1}^{\infty} a_n \int_0^{T_0} \cos n\omega_0 t dt \\ &+ \sum_{n=1}^{\infty} b_n \int_0^{T_0} \sin n\omega_0 t dt \end{aligned}$$

- ◆ 2<sup>nd</sup> and 3<sup>rd</sup> terms integrates to zero over one period of time. Therefore only the first term survives:

$$\int_0^{T_0} x(t) dt = a_0 \int_0^{T_0} dt = a_0 T_0$$

- ◆ Therefore

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt$$

# How to find $a_n$ and $b_n$ coefficients? (1)

---

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- ◆ To determine  $a_0$ , we simply integrate both sides of the equation over one period  $T_0$ :

$$\int_0^{T_0} x(t) \cos m\omega_0 t dt = a_0 \int_0^{T_0} \cos m\omega_0 t dt$$

$$+ \sum_{n=1}^{\infty} a_n \int_0^{T_0} \cos n\omega_0 t \cos m\omega_0 t dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_0^{T_0} \sin n\omega_0 t \cos m\omega_0 t dt$$

- ◆ But:

$$\int_0^{T_0} \cos m\omega_0 t dt = 0 \quad \text{and} \quad \int_0^{T_0} \cos n\omega_0 t \cos m\omega_0 t dt = 0 \quad \text{if } n \neq m$$

- ◆ When  $n = m$ ,

$$\int_0^{T_0} \cos m\omega_0 t \cos m\omega_0 t dt = \frac{T_0}{2}$$

## How to find $a_n$ and $b_n$ coefficients? (2)

---

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- ◆ Therefore, the ONLY term that survives after multiply by  $\cos m\omega_0 t$  and integration is:

$$\int_0^{T_0} x(t) \cos m\omega_0 t dt = a_m \frac{T_0}{2}$$

- ◆ Hence,  $a_n = \frac{2}{T_0} \int_0^{T_0} x(t) \cos n\omega_0 t dt$  ( $m = n$ )

- ◆ Similarly to find  $b_n$  multiply  $x(t)$  by  $\sin m\omega_0 t$  and integration over  $T_0$ :

$$\int_0^{T_0} x(t) \sin m\omega_0 t dt = b_m \frac{T_0}{2}$$

- ◆ Hence,  $b_n = \frac{2}{T_0} \int_0^{T_0} x(t) \sin n\omega_0 t dt$

# Compact form of Fourier Series

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$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- ◆ A more compact form of the Fourier Series is derived with the trigonometric identity:

$$\begin{aligned} C \cos(\omega_0 t + \theta) &= C \cos \theta \cos \omega_0 t - C \sin \theta \sin \omega_0 t \\ &= a \cos \omega_0 t + b \sin \omega_0 t \end{aligned}$$

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \\ &= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \end{aligned}$$

where

$$C_0 = a_0$$

DC term

$$C_n = \sqrt{a_n^2 + b_n^2}$$

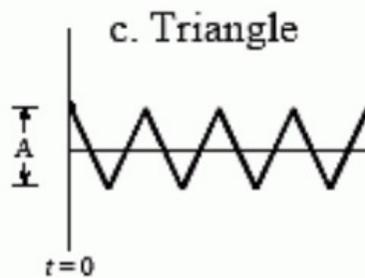
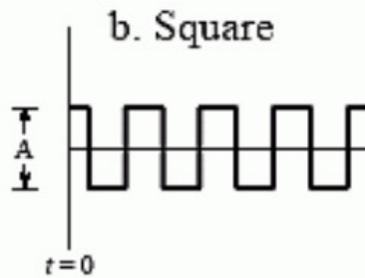
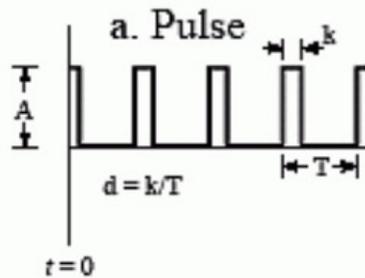
amplitude

$$\theta_n = \tan^{-1} \left( \frac{b_n}{a_n} \right)$$

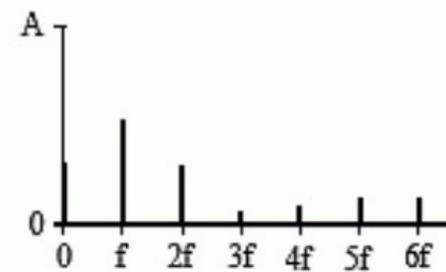
phase angle

# Fourier Series of common signals (1)

## Time Domain



## Frequency Domain

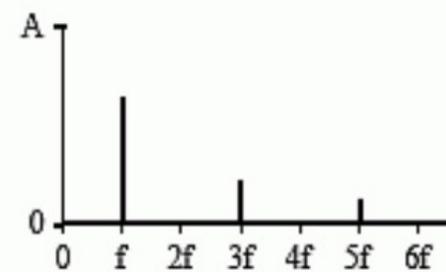


$$a_0 = A d$$

$$a_n = \frac{2A}{n\pi} \sin(n\pi d)$$

$$b_n = 0$$

( $d = 0.27$  in this example)

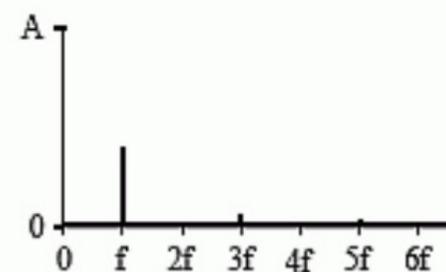


$$a_0 = 0$$

$$a_n = \frac{2A}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$b_n = 0$$

(all even harmonics are zero)



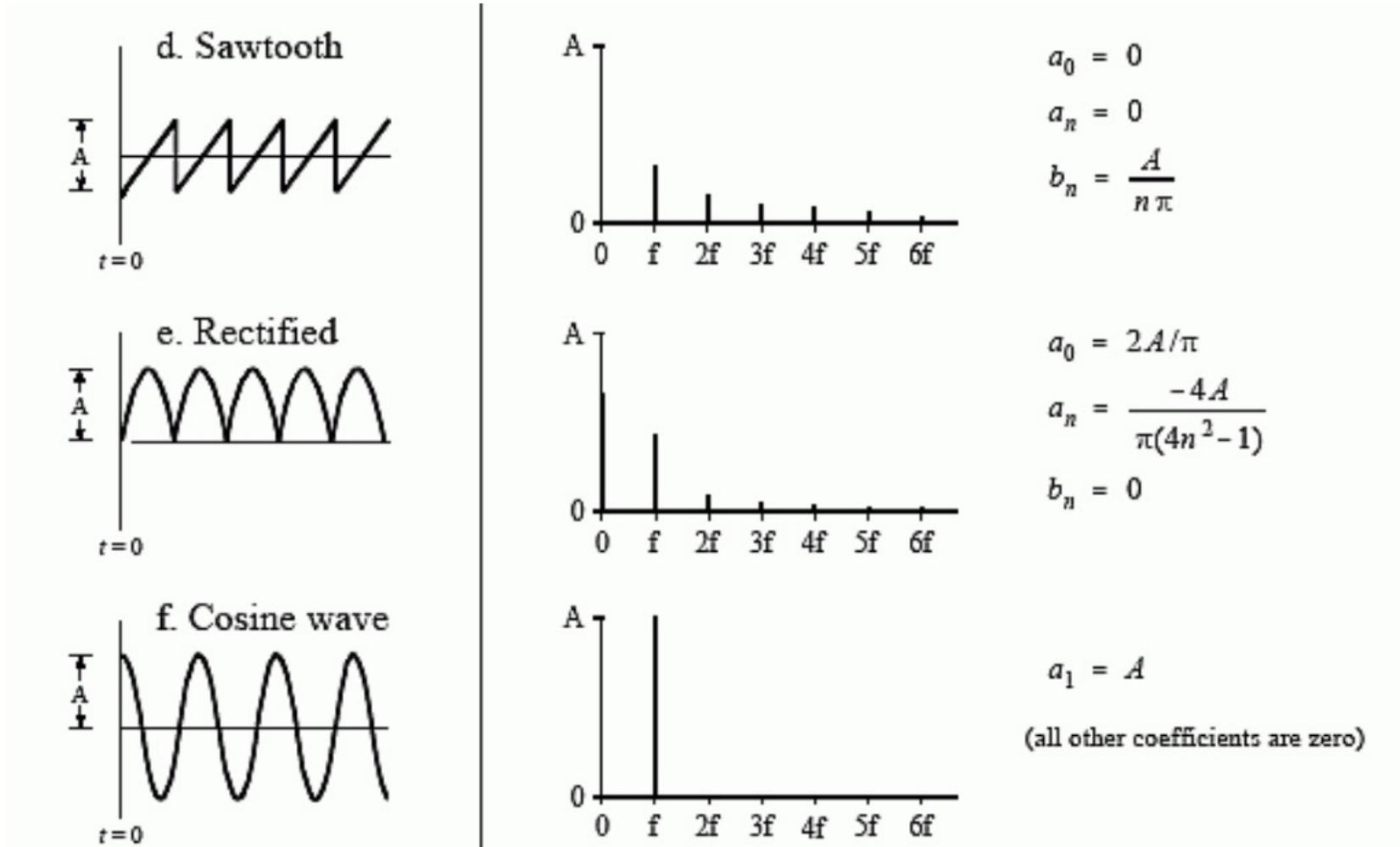
$$a_0 = 0$$

$$a_n = \frac{4A}{(n\pi)^2}$$

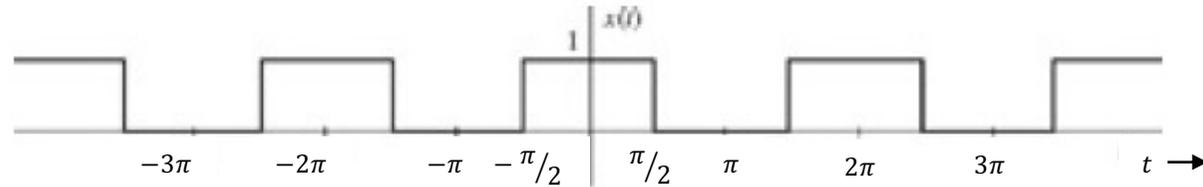
$$b_n = 0$$

(all even harmonics are zero)

# Fourier Series of common signals (2)



# Fourier series of an even signal

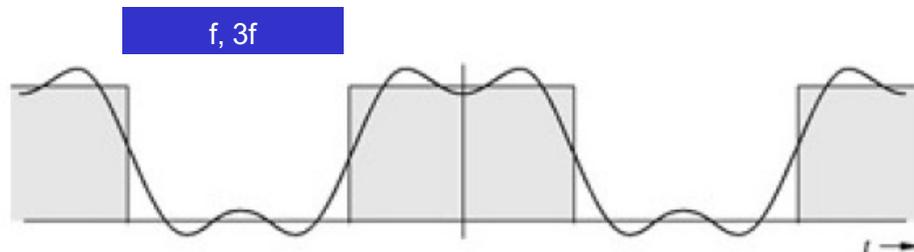
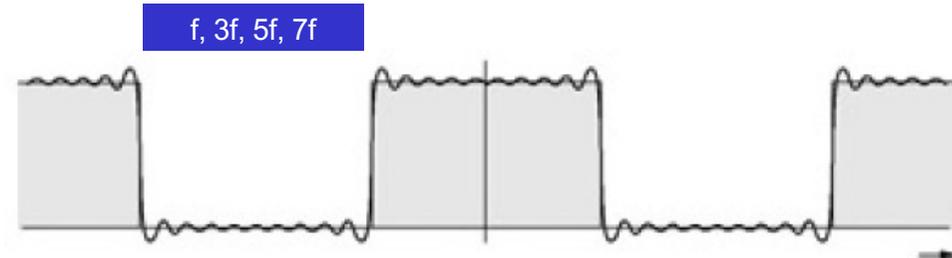
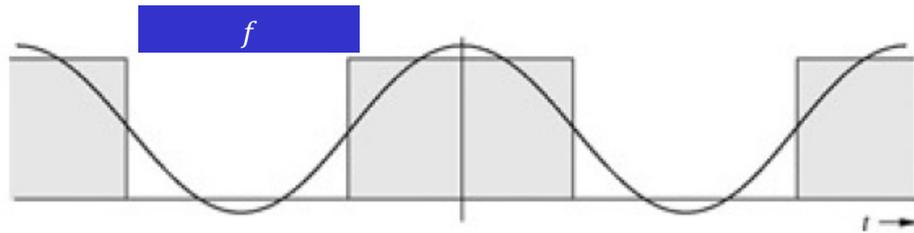
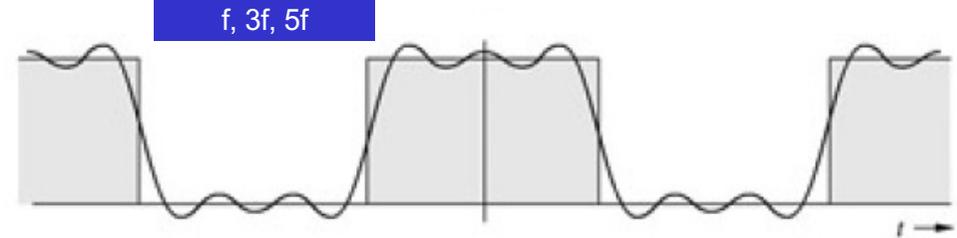
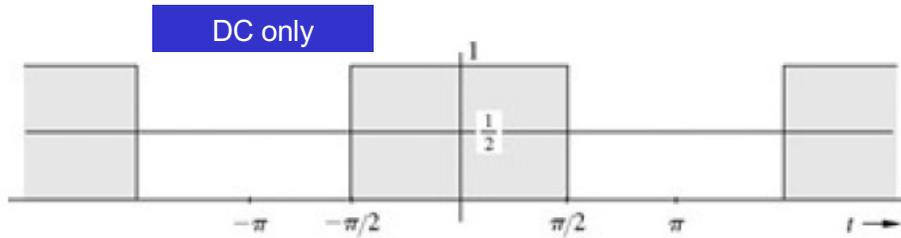
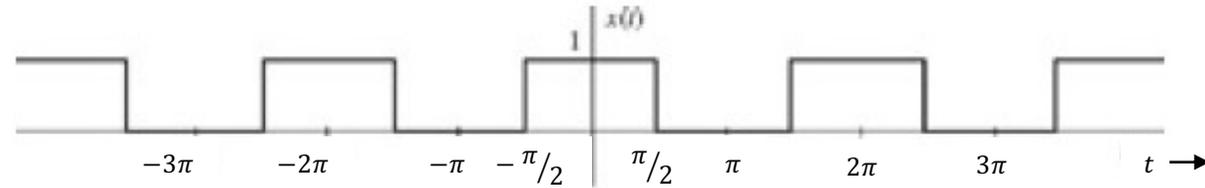


- ◆ The Fourier series for the square-pulse periodic signal shown above is:

$$x(t) = \frac{1}{2} + \frac{2}{\pi} \left( \cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t - \frac{1}{7} \cos 7t + \dots \right)$$

- ◆ The symmetry of this even signal result in three properties:
  1. Such symmetry implies an even function. Therefore the Fourier series representation only has cosine terms which are also even functions.
  2. This symmetry at  $t = 0$  also result in phase angle at all harmonic frequencies = 0.
  3. It only has odd harmonic components – no even harmonic components.

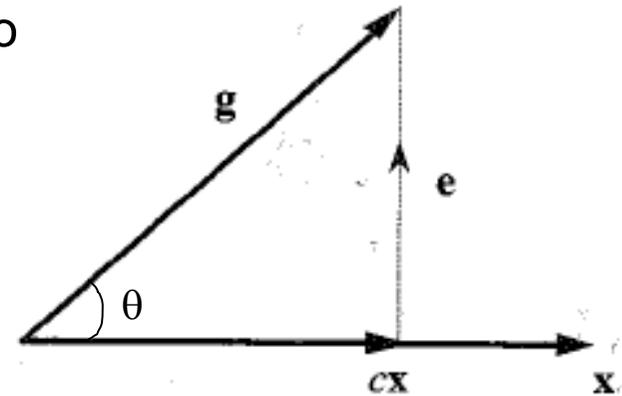
# Fourier coefficients and waveshaping



- ◆ Low frequencies determines overall shape
- ◆ High frequencies determines detail structures

# A Vector view of Signal

- ◆ To understand why a signal can be represented by linear sum of sinusoidal waveforms, it is useful to consider electrical signals as VECTORS.
- ◆ A vector is specified by its magnitude (or length) and its direction.
- ◆ Consider two vectors  $\mathbf{g}$  and  $\mathbf{x}$ . If we project  $\mathbf{g}$  onto  $\mathbf{x}$ , we get  $c\mathbf{x}$ , where  $c$  is a scalar (i.e. constant with no direction).
- ◆ If we approximate  $\mathbf{g}$  with  $c\mathbf{x}$ , then
$$\mathbf{g} = c\mathbf{x} + \mathbf{e}$$
- ◆  $\mathbf{e}$ , the error vector, is minimum when it is perpendicular to  $\mathbf{x}$ .
- ◆  $c\mathbf{x}$  is known as the **projection** of  $\mathbf{g}$  onto  $\mathbf{x}$ .
- ◆ It can be shown (in the notes below) that:



Dot product

$$\mathbf{g} \cdot \mathbf{x} = |\mathbf{g}||\mathbf{x}| \cos \theta$$

$$c = \frac{\mathbf{g} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{1}{|\mathbf{x}|^2} \mathbf{g} \cdot \mathbf{x}$$

# Orthogonal Set of signals

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- ◆ If vector  $\mathbf{g}$  is at right angle to vector  $\mathbf{x}$ , then the projection of  $\mathbf{g}$  and  $\mathbf{x}$  is zero. These two vectors (or signals) are known to be **orthogonal**.
- ◆ It can easily be shown that two sinusoidal signals of DIFFERENT frequencies are orthogonal to each other.
- ◆ The complete set of sinusoidal signals (i.e. of all possible frequency) forms a COMPLETE orthogonal set of signals.
- ◆ What this means is that ALL time domain signals can be formed out of projects (or components) onto these these sinusoidal set of signals!
- ◆ This is the foundation of **Fourier Series** and **Fourier Transform**, which will be discussed further at the next Lecture.

# Three Big Ideas

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1. **Time domain** view of a signal is often insufficient. It is often more informative to consider how the signal would appear as a function of frequency, in the **frequency domain**.
2. Any time varying signal can be **decomposed into sinusoidal constituent components** of specific frequencies, phases, and amplitudes, just like the tidal level. This is the main idea of Fourier.
3. Two sinusoidal signals of different frequencies are **orthogonal** to each other, meaning that they have nothing in common, and it is not possible to “produce” one from the other through any linear methods.