Imperial College London

Lecture 3

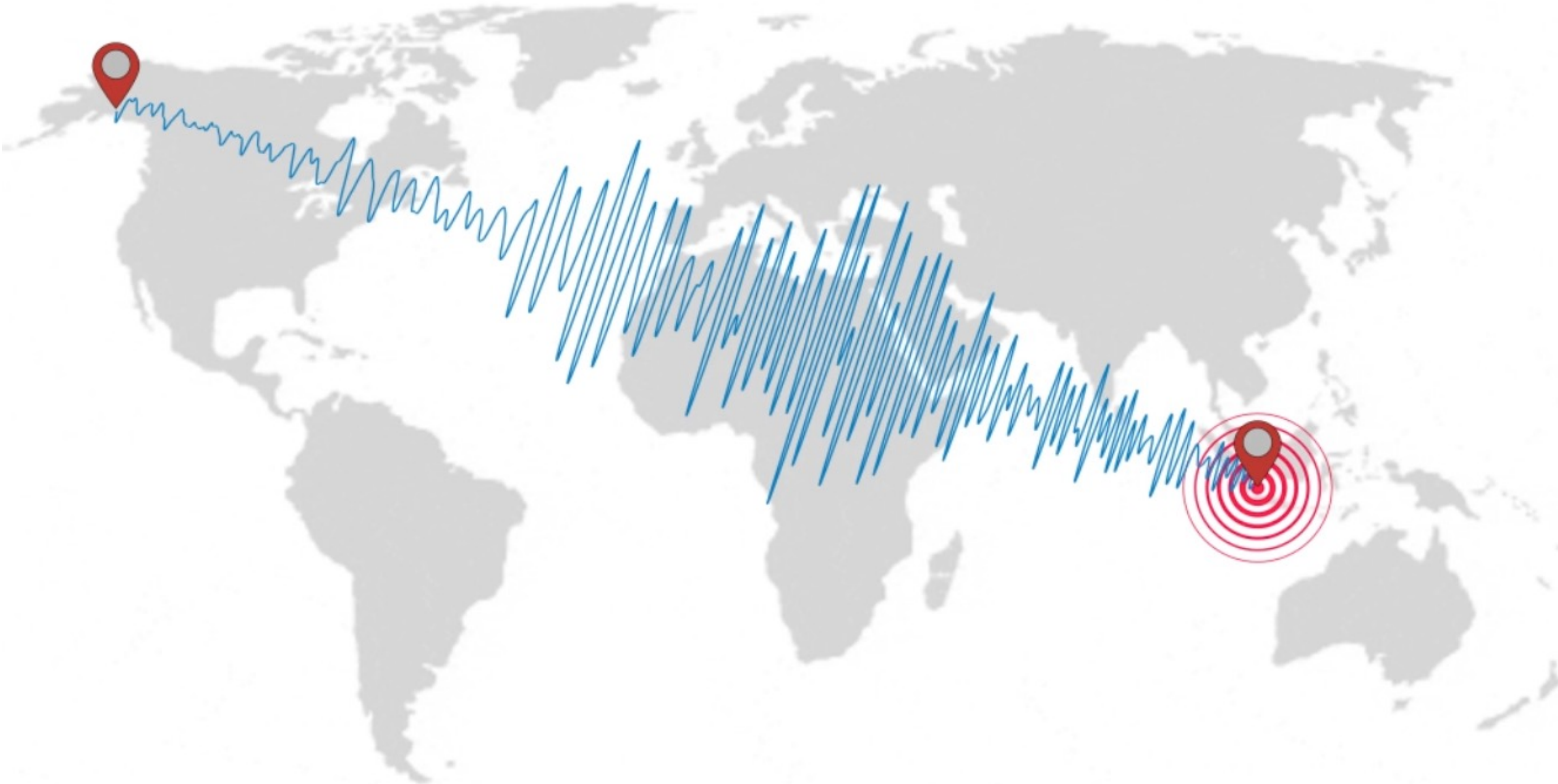
Frequency Domain view of signals

Peter Y K Cheung

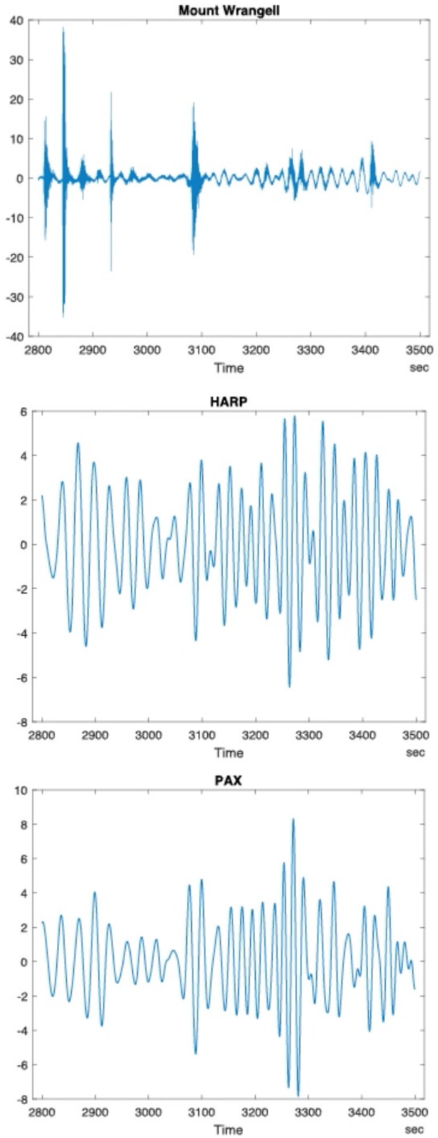
Dyson School of Design Engineering
Imperial College London

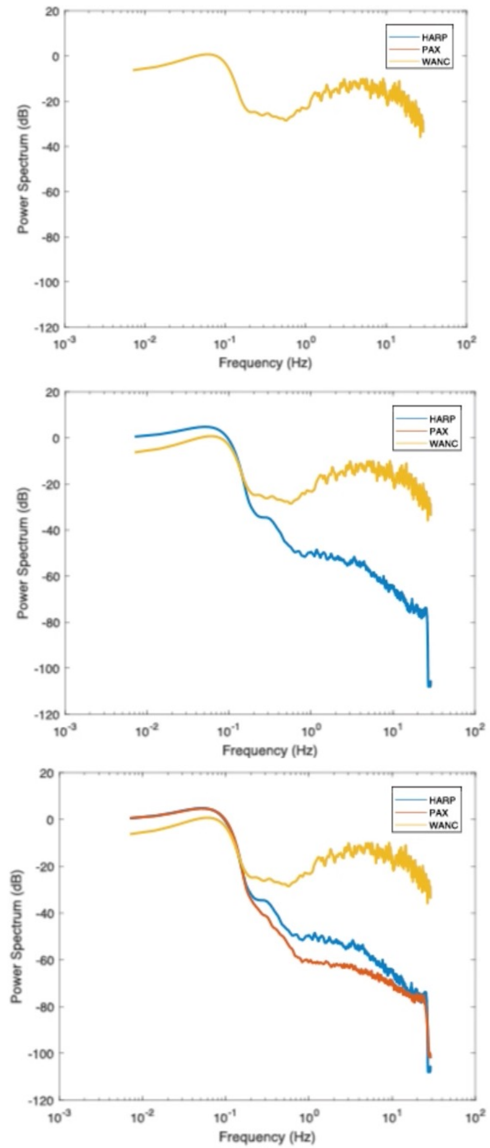
URL: www.ee.ic.ac.uk/pcheung/teaching/DE2_EE/
E-mail: p.cheung@imperial.ac.uk

Example – Seimic signal measured in Alaska due to Sumatra Earthquake in 2004

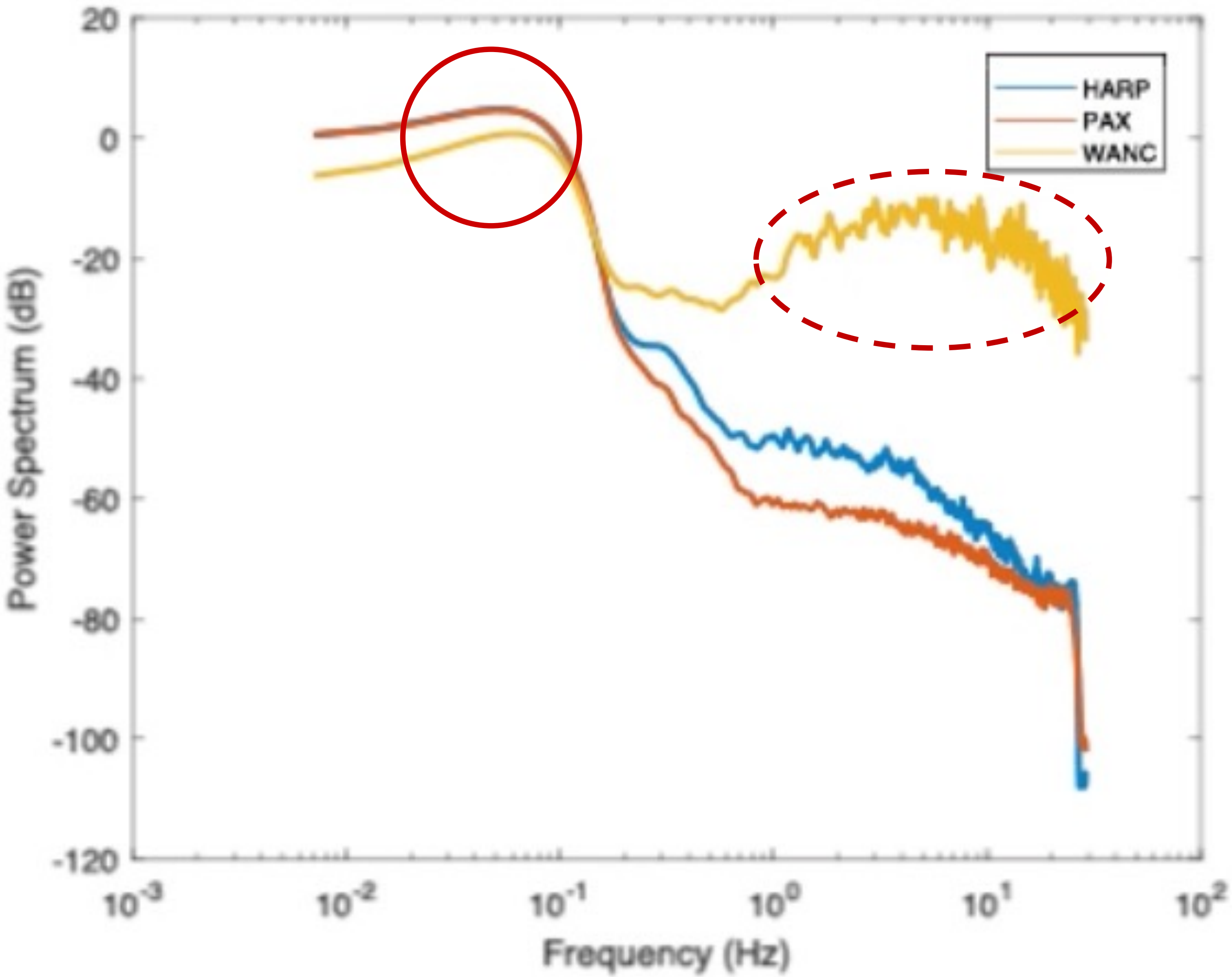


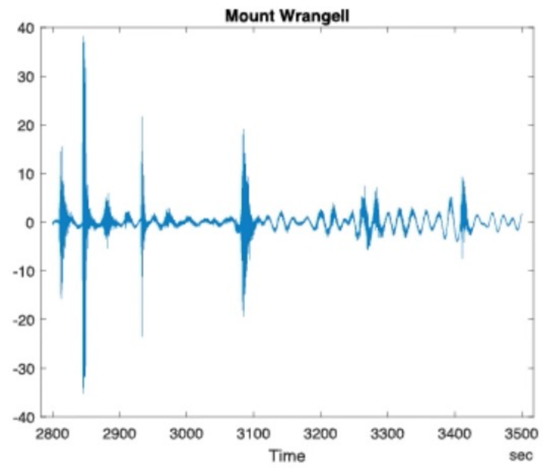
Three stations recorded data for analysis



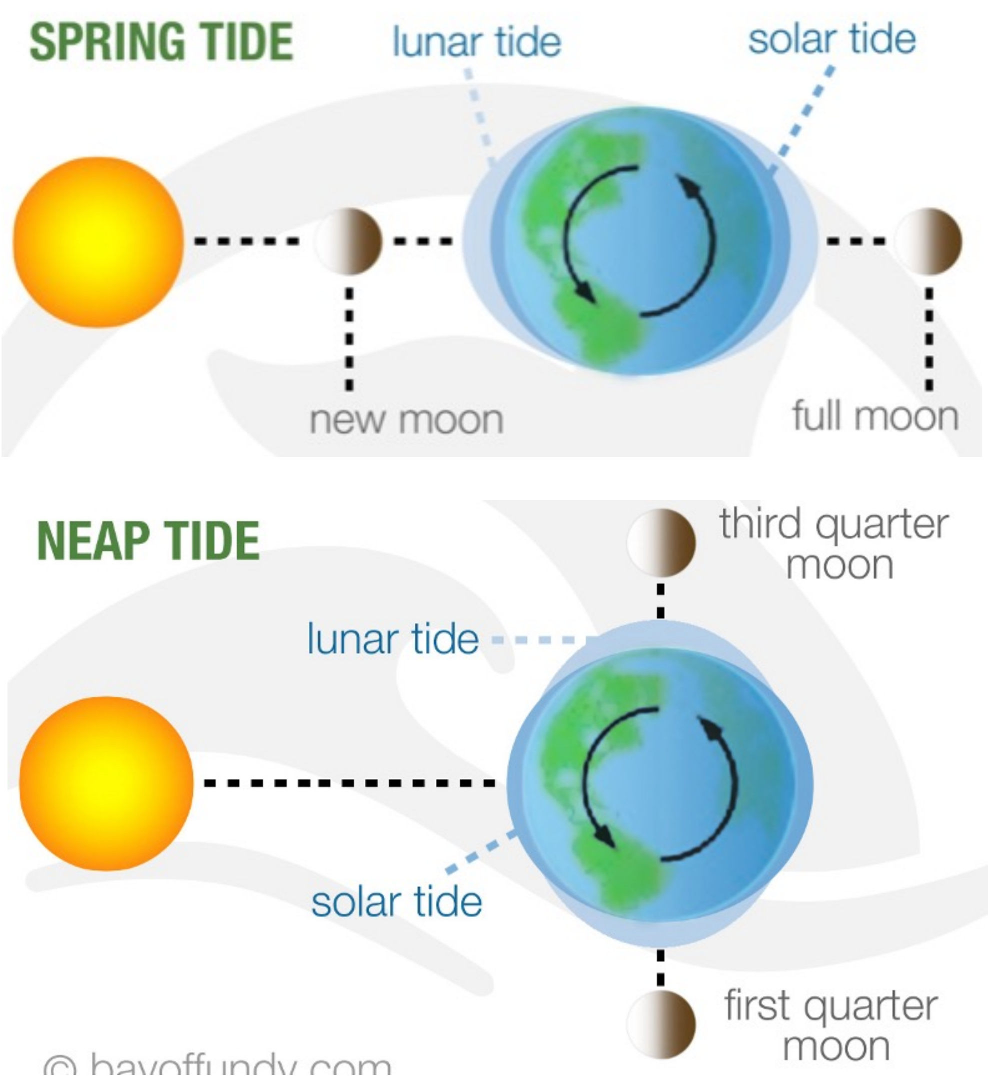


Frequency domain view of signal – more informative





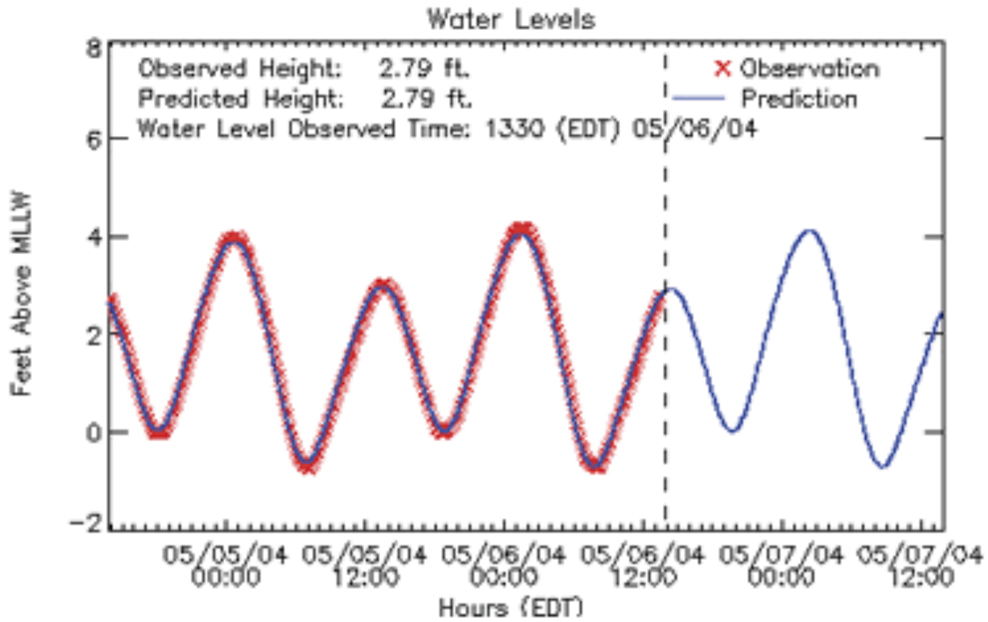
Prediction of Tides

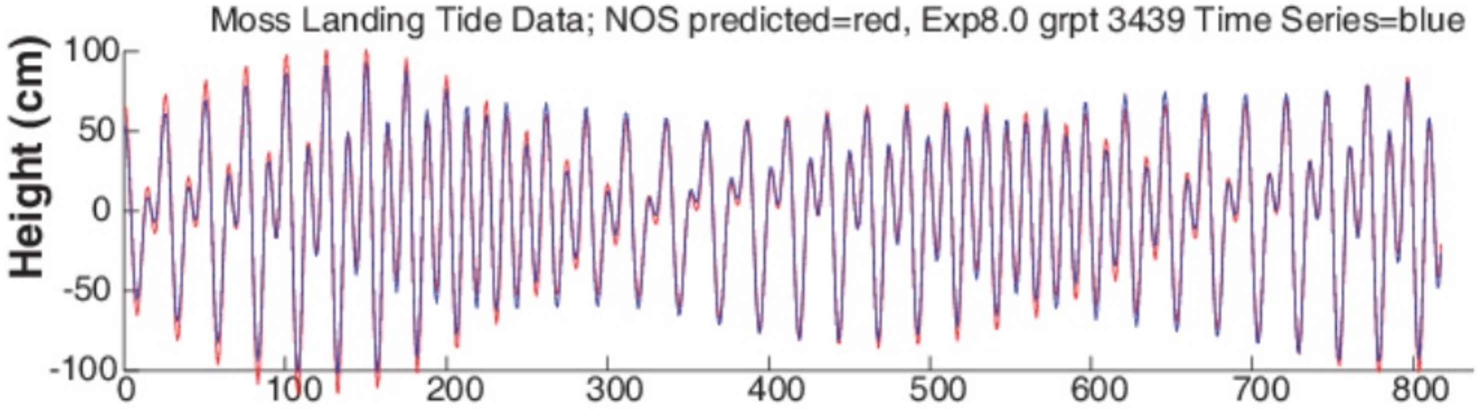


- Early scientists:
  - Isaac Newton
  - Joseph Fourier
  - William Thomson (Lord Kelvin)
  - Pierre-Simon Laplace

http://oceanmotion.org/html/background/tides-observing.htm

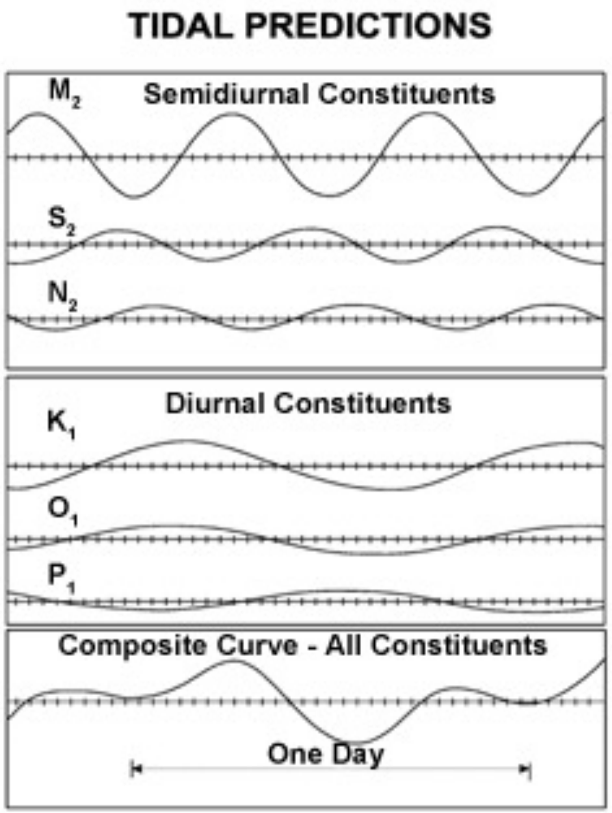
Periodicity of the Tides

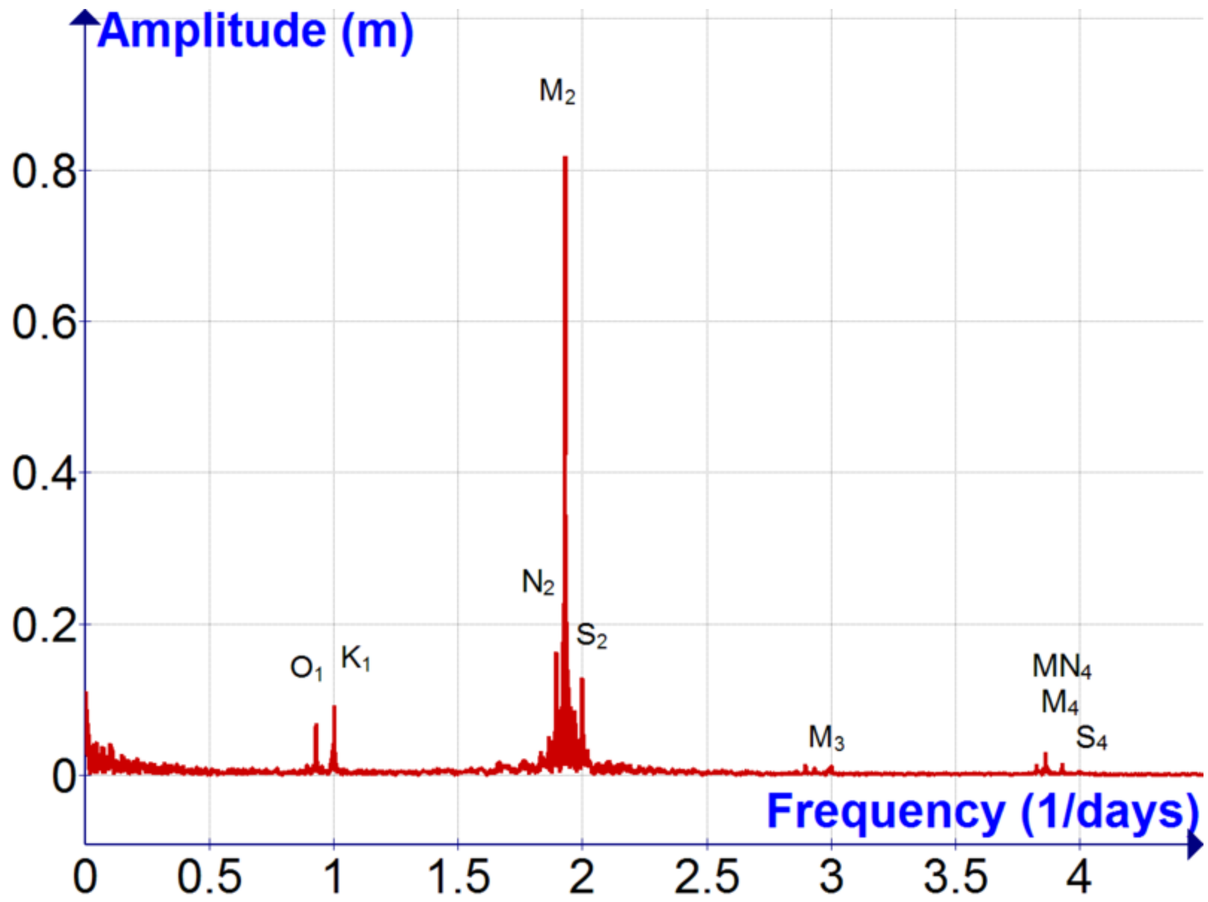




http://oceanmotion.org/html/background/tides-observing.htm

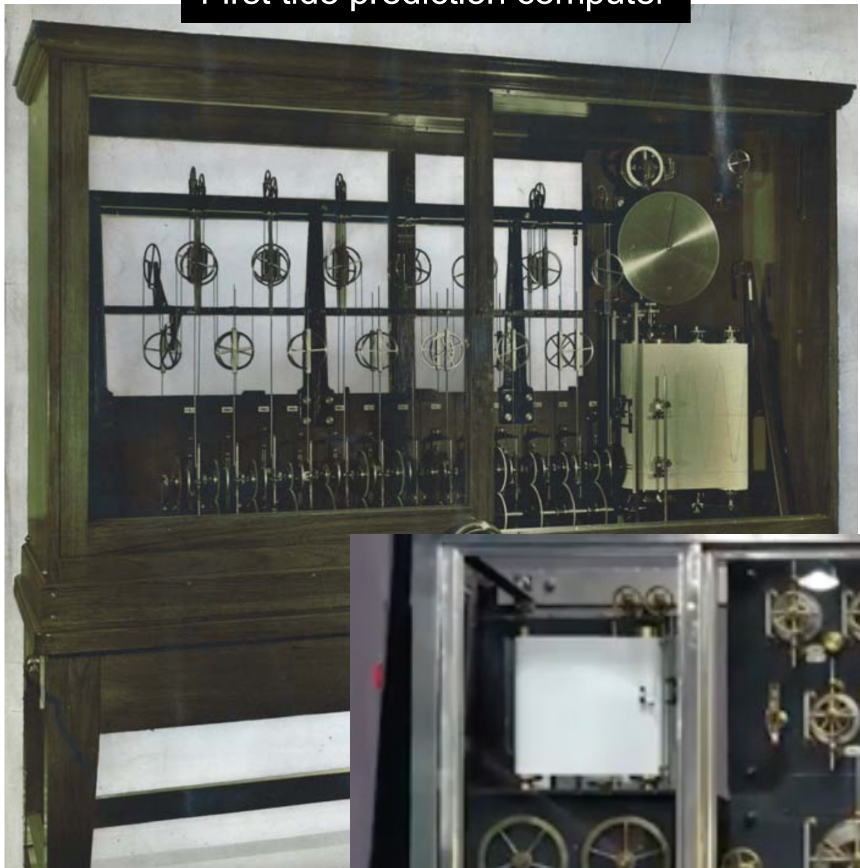
Tides decomposed into periodic constituents





Kelvin’s Tide Prediction Machine

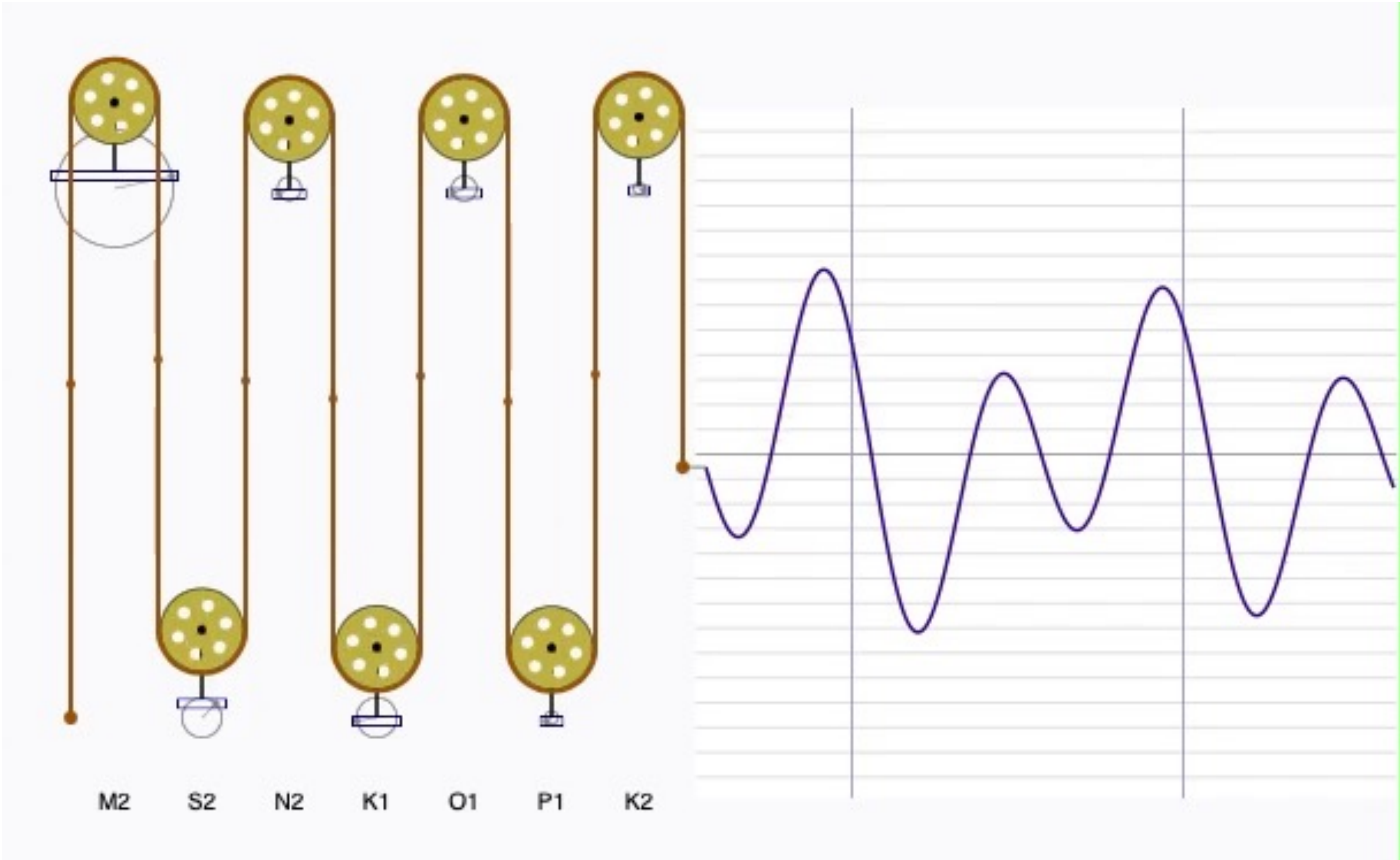
First tide prediction computer



William Thomson (Lord Kelvin)

Doodson_lege machine in Liverpoool

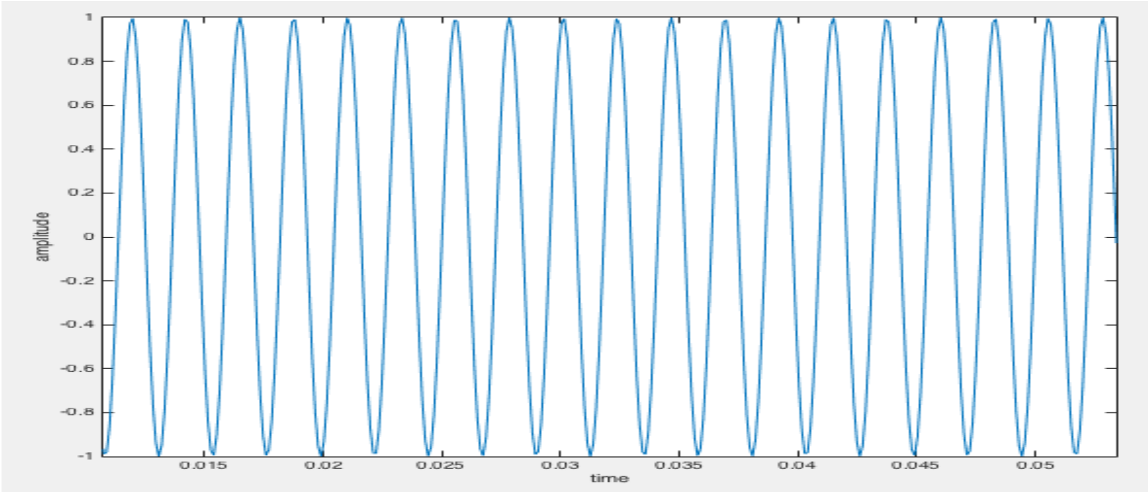
Animation of the Tide Prediction Machine



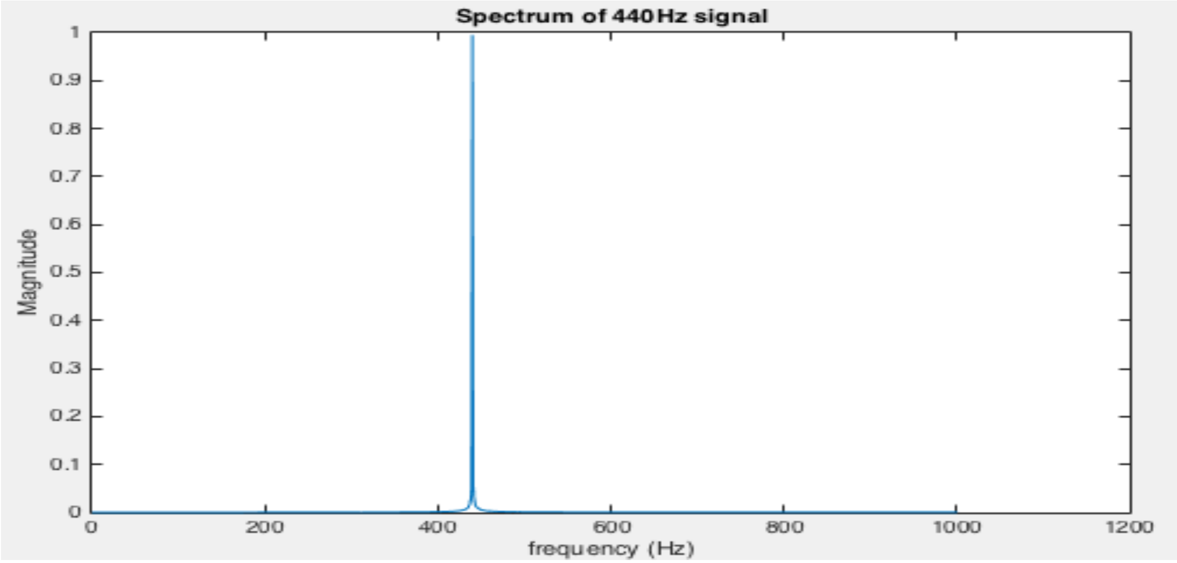
https://www.ams.org/publicoutreach/feature-column/fcarc-tidesiii3

Time vs Frequency view of a sinewave

- Sinewave (sinusoidal signal) in time domain

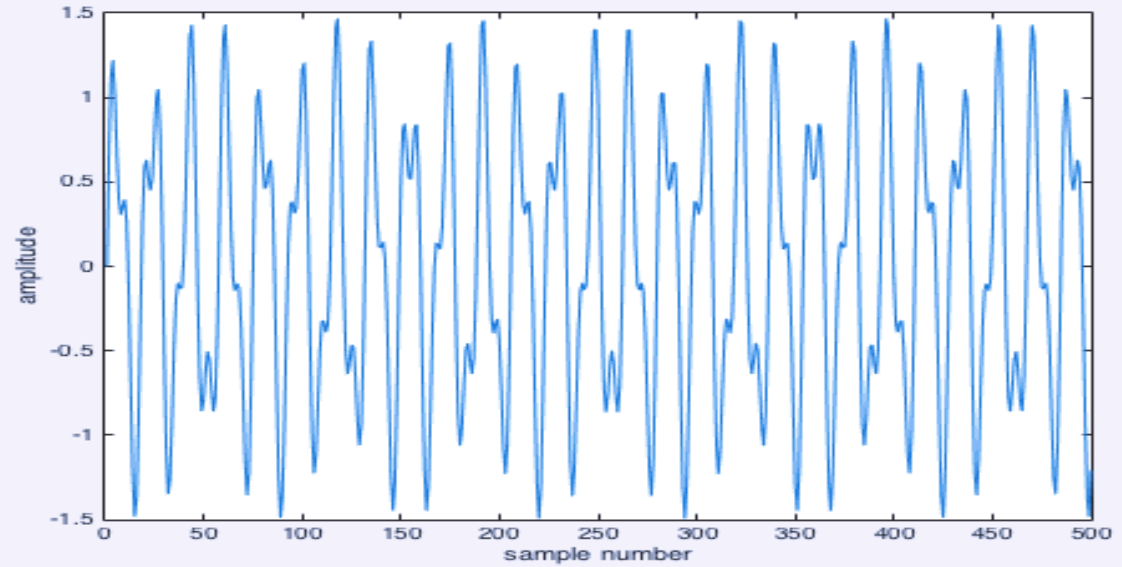


- Same sinewave in frequency domain

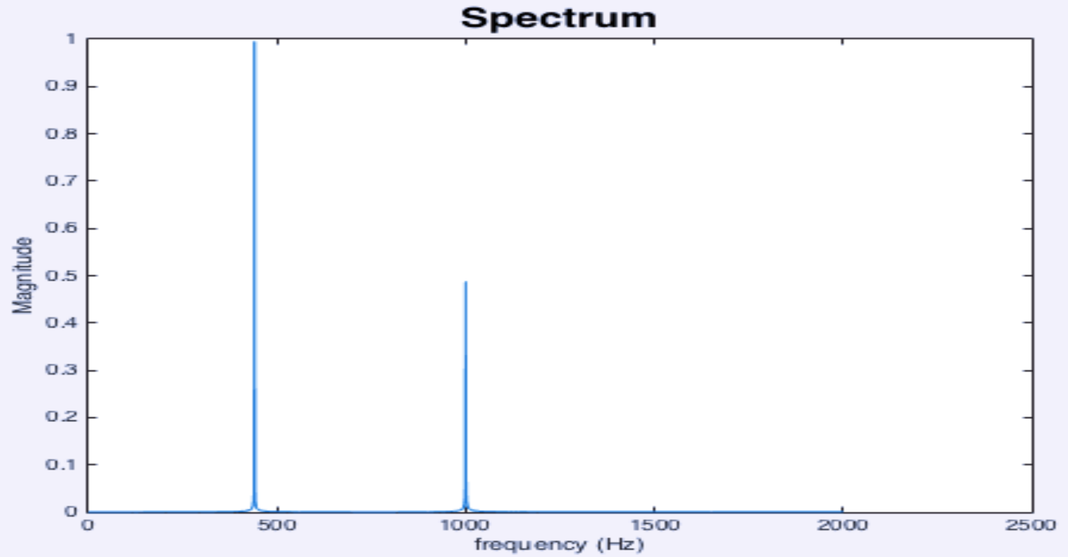


Two sinewaves

- Adding 440Hz to 1kHz signal. The 440Hz is twice as large as the 1kHz signal.

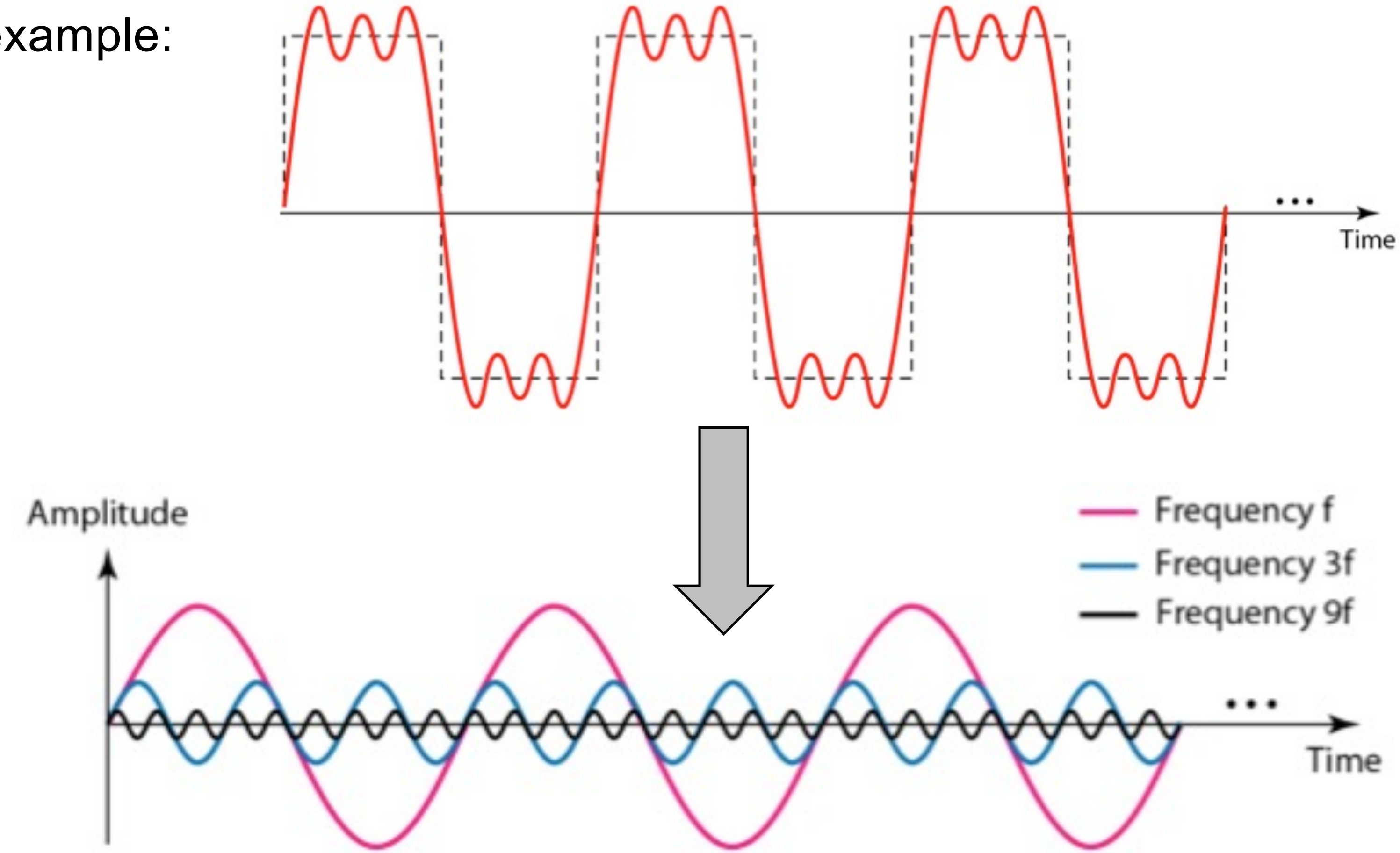


- Spectrum of two sinewaves



Key idea – Fourier's theory

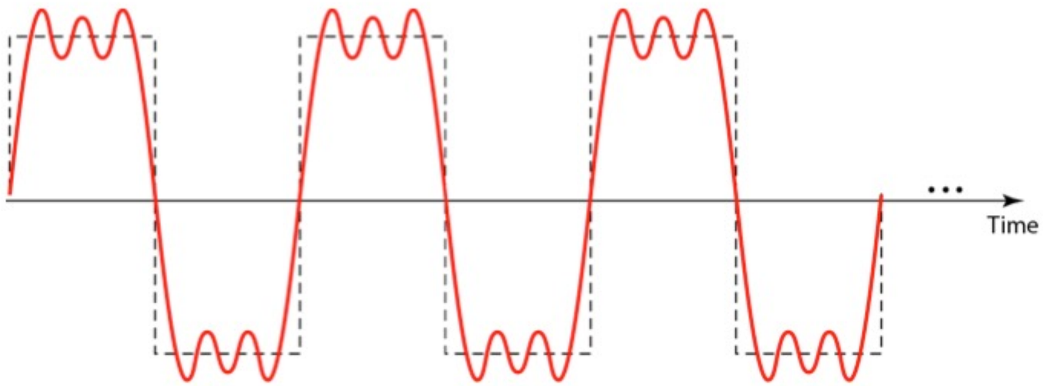
- Basic idea – **any time domain signal can be constructed from weighted linear sum of sinusoidal signals** (sine or cosine signals) at different frequencies.
- For example:



Spectrum – Frequency domain representation

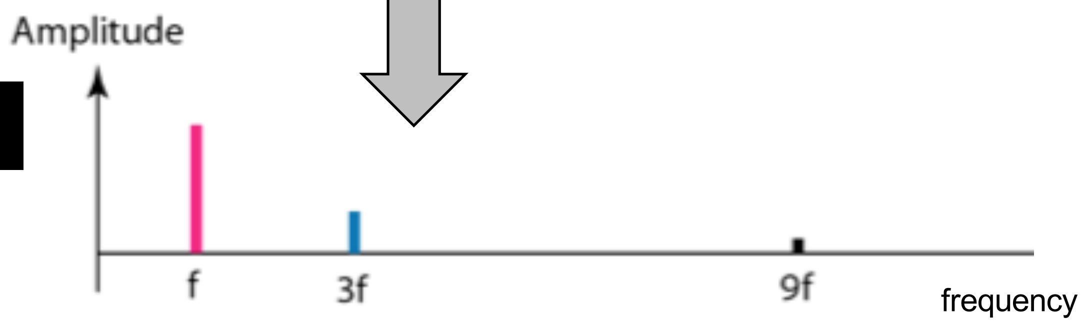
- Instead of having to store individual time samples, we only need to store the amplitude, frequency and phase of each sinusoidal signal.

Time domain



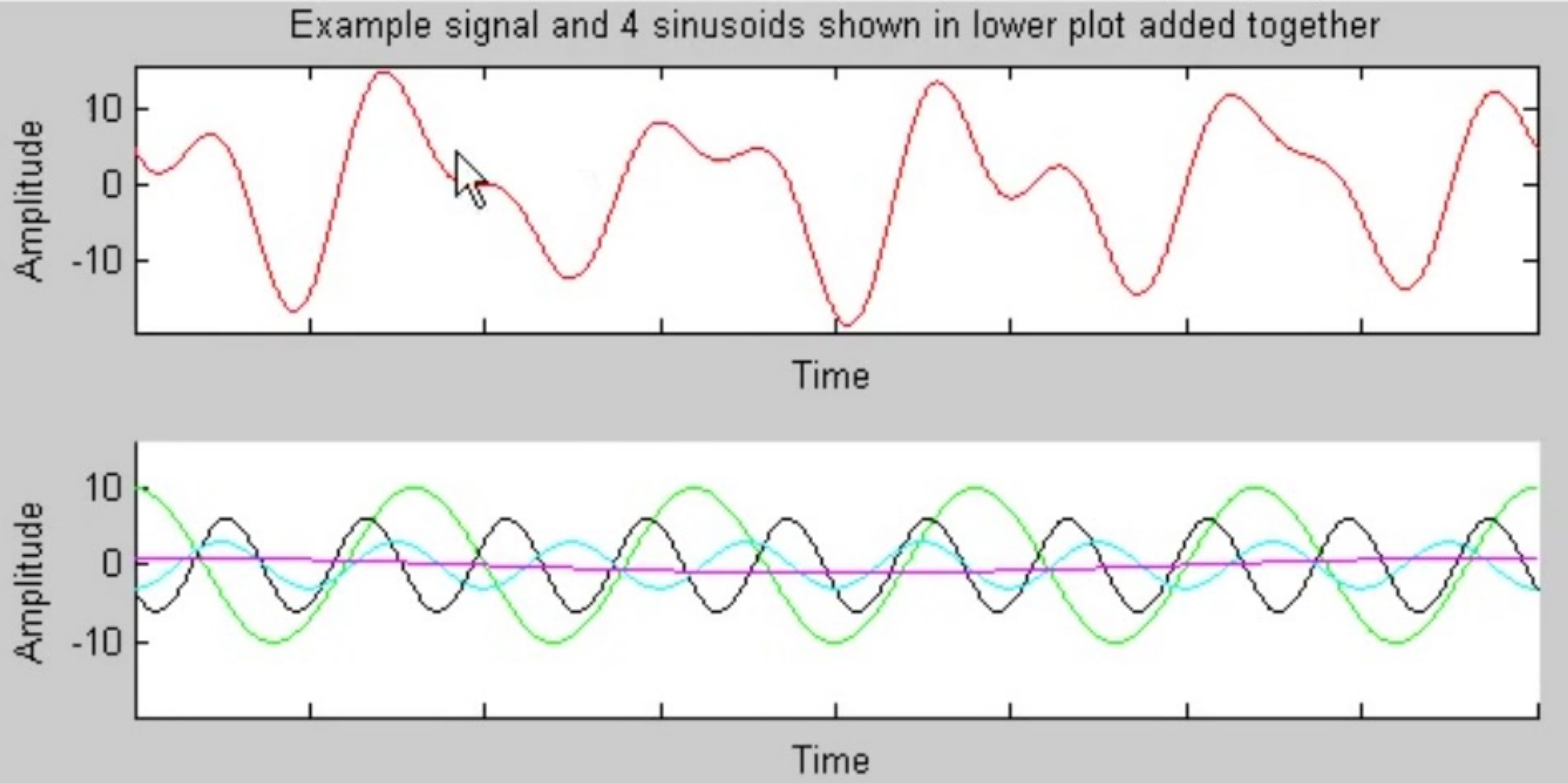
- Spectrum of signal in frequency domain is represented by amplitude value for each frequency. There is also phase vs frequency, which is not shown here.

Frequency domain



Another Example

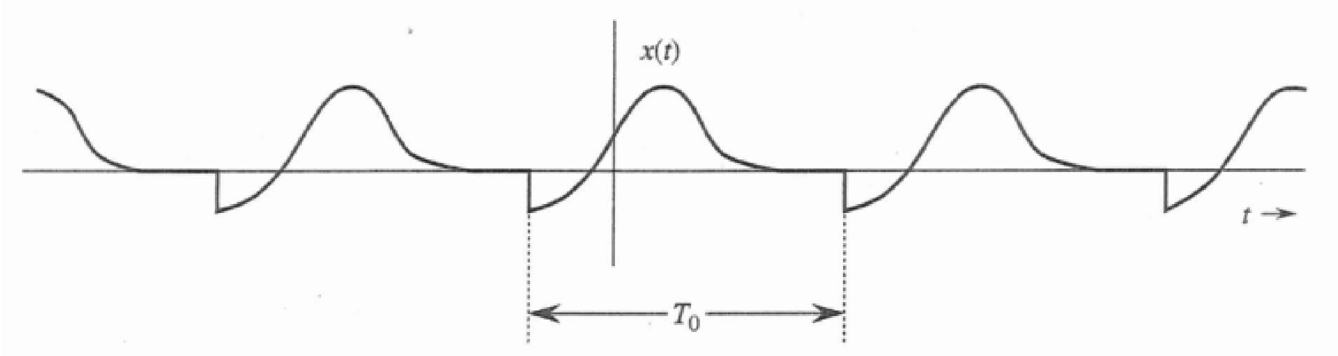
- Here is another time domain signal that is constructed with four sine waves:



Periodic Signal & Fourier Series

- A periodic signal $x(t)$ with a period of To has the property:

$$x(t) = x(t + T_0) \quad \text{for all } t$$



- **Fourier series** expresses $x(t)$ as a **weighted linear sum of sinusoids** (or expontentials) of the fundamental frequency f_0 = 1/T_0 and all it harmonics nf_0 where n = 2, 3, 4…..

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad \text{for all integers n}$$

- ω_0 is called the **fundamental frequency** such that (f_0 in cycles/sec or Hz, ω_0 in radians/sec)

$$\omega_0 = 2\pi f_0 = 2\pi / T_0 \quad \text{and} \quad n\omega_0 \text{ are the harmonic frequencies}$$

- a_0 is the DC (mean) value of $x(t)$ and a_n , b_n are the Fourier coefficients at the frequency $n\omega_0$

How to find a_0?

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ )$$

- To determine a_0, we multiply both sides by $\cos m\omega_0 t$ and intergrate over T_0:

$$\int_0^{T_o} x(t)dt = a_0 \int_0^{T_o} dt$$

$$+ \sum_{n=1}^{\infty} a_n \int_0^{T_o} \cos n\omega_0 t\ dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{T_o}^{T_o} \sin n\omega_0 t\ dt$$

- 2nd and 3rd terms integrates to zero over one period of time. Therefore only the first term survives:

$$\int_0^{T_o} x(t)dt = a_0 \int_0^{T_o} dt = a_0\ T_o$$

- Therefore

$$a_0 = \frac{1}{T_o}\int_0^{T_o} x(t)dt$$

How to find a_n and b_n coefficients? (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ )$$

- To determine a_0, we simply intergrate both sides of the equation over one period T_0:

$$\int_0^{T_o} x(t) \cos m\omega_0 t\, dt = a_0 \int_0^{T_o} \cos m\omega_0 t\, dt$$
$$+ \sum_{n=1}^{\infty} a_n \int_0^{T_o} \cos n\omega_0 t\ \cos m\omega_0 t\, dt$$
$$+ \sum_{n=1}^{\infty} b_n \int_{T_o}^{T_o} \sin n\omega_0 t\ \cos m\omega_0 t\, dt$$

- But:

$\int_0^{T_o} \cos m\omega_0 t\, dt$ = 0 and $\int_0^{T_o} \cos n\omega_0 t\ \cos m\omega_0 t\, dt$ = 0 if n ≠ m

- When n = m,

$$\int_0^{T_o} \cos m\omega_0 t\ \cos m\omega_0 t\, dt = \frac{T_0}{2}$$

How to find a_n and b_n coefficients? (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ )$$

- Therefore, the ONLY term that survives after multiply by $\cos m\omega_0 t$ and integration is:

$$\int_0^{T_o} x(t) \cos m\omega_0 t \, dt = a_m \frac{T_0}{2}$$

- Hence, $a_n = \frac{2}{T_0}\int_0^{T_o} x(t) \cos n\omega_0 t \, dt$ (m = n)
- Similarly to find b_n multiply x(t) by $\sin m\omega_0 t$ and integration over T_o:

$$\int_0^{T_o} x(t) \sin m\omega_0 t \, dt = b_m \frac{T_0}{2}$$

- Hence, $b_n = \frac{2}{T_0}\int_0^{T_o} x(t) \sin n\omega_0 t \, dt$

Compact form of Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ )$$

- A more compact form of the Fourier Series is derived with the trigonometric identity:

$$C\cos(\omega_0 t + \ \theta) = C\cos\theta\cos\omega_0 t\ -\ C\sin\theta\sin\omega_0 t$$
$$= a\cos\omega_0 t + b\sin\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\ )$$
$$= C_0 + \sum_{n=1}^{\infty}\ C_n\cos(\ n\omega_0 t + \ \theta_n)$$

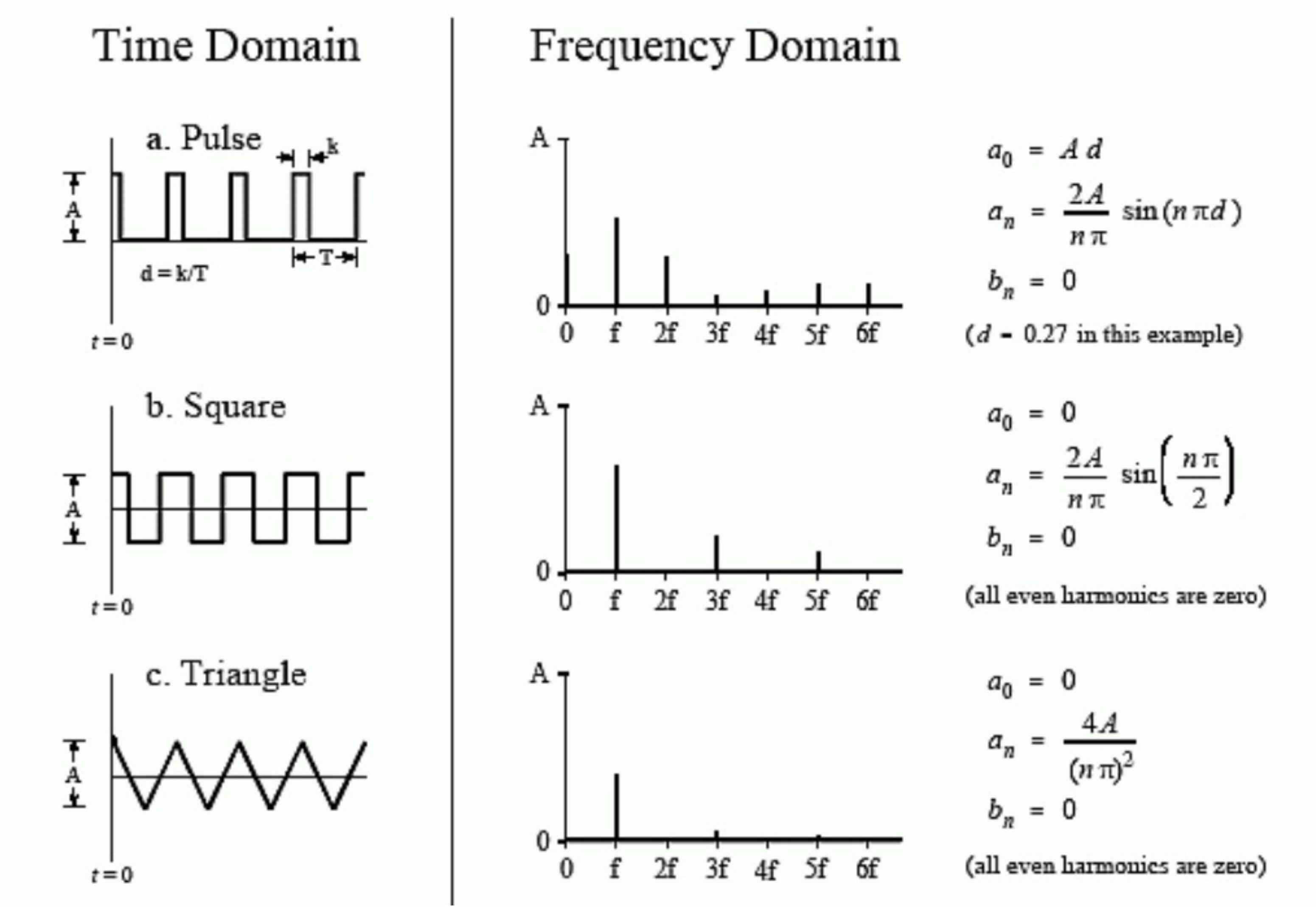
where

$$C_0\ = a_0$$ DC term

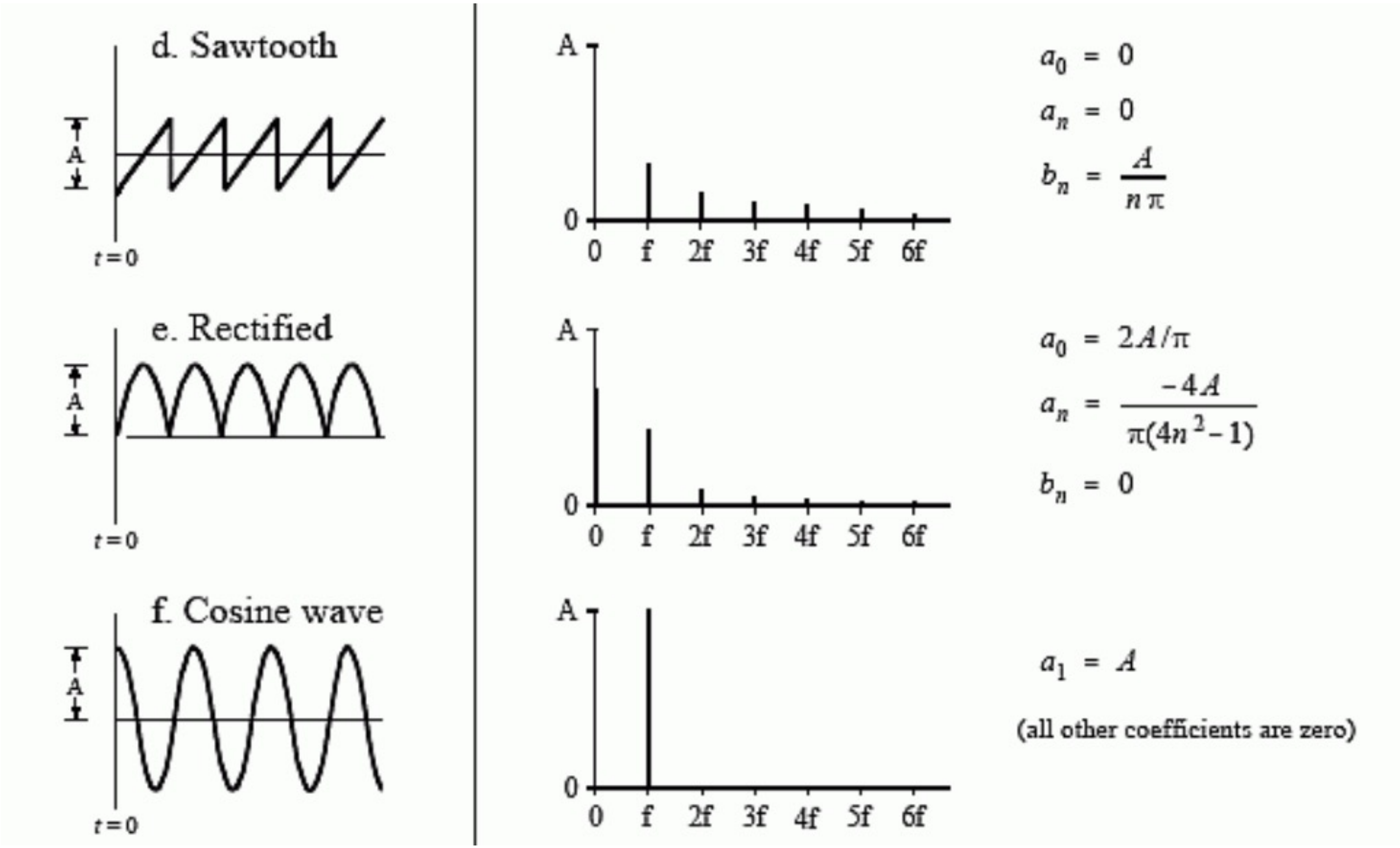
$$C_n\ = \ \sqrt{{a_n}^2 + \ {b_n}^2}$$ amplitude

$$\theta_n\ = \ tan^{-1} - \left(\frac{b_n}{a_n}\right)$$ phase angle

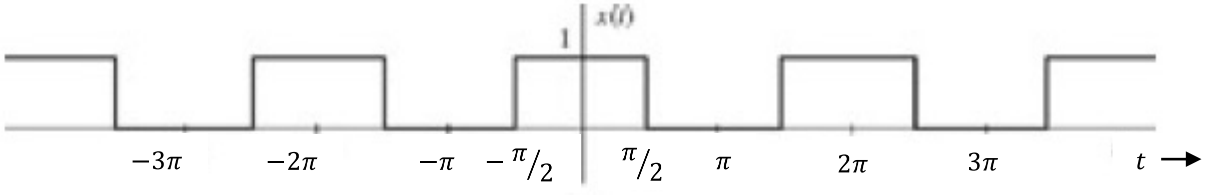
Fourier Series of common signals (1)



Fourier Series of common signals (2)



Fourier series of an even signal



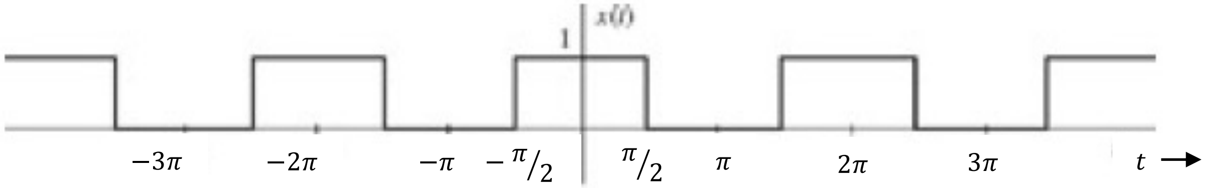
- The Fourier series for the square-pulse periodic signal shown above is:

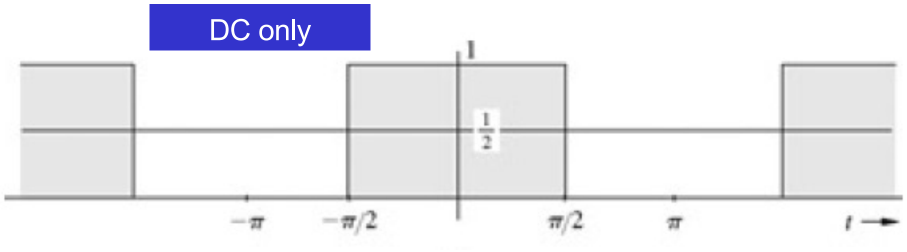
$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t \ + \ \ldots\ldots\right)$$

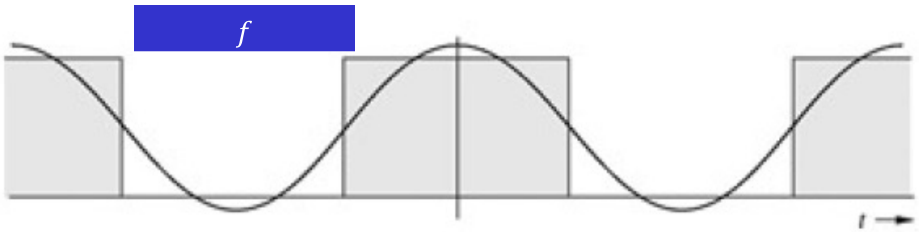
- The symmetry of this even signal result in three properties:

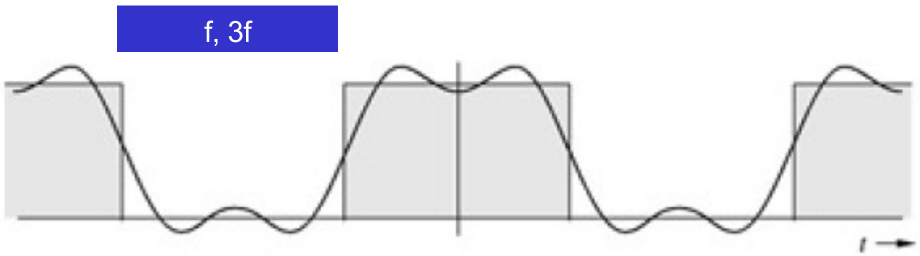
1. Such symmetry implies an even even function. Therefore the Fourier series representation only has cosine terms which are also even functions.
2. This symmetry at t = 0 also result in phase angle at all harmonic frequencies = 0.
3. It only has odd harmonic components – no even harmonic components.

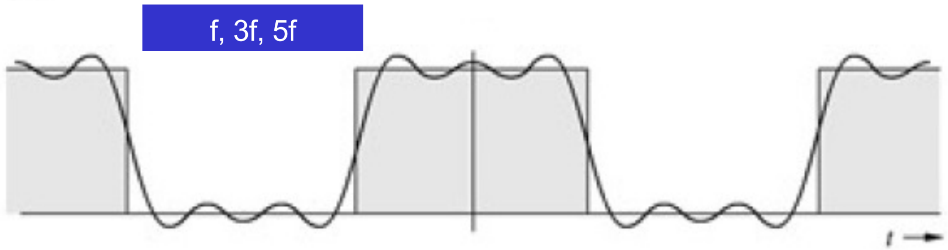
Fourier coefficiences and waveshaping

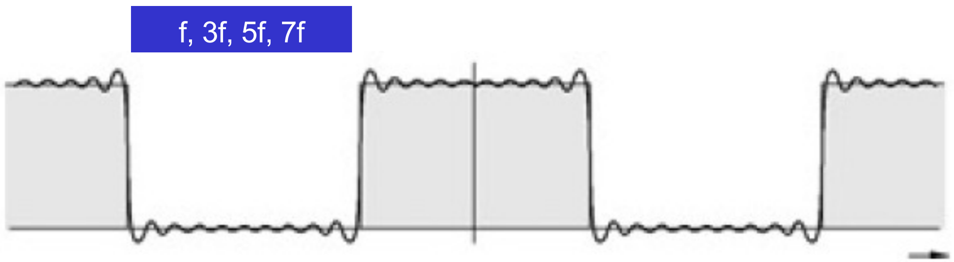










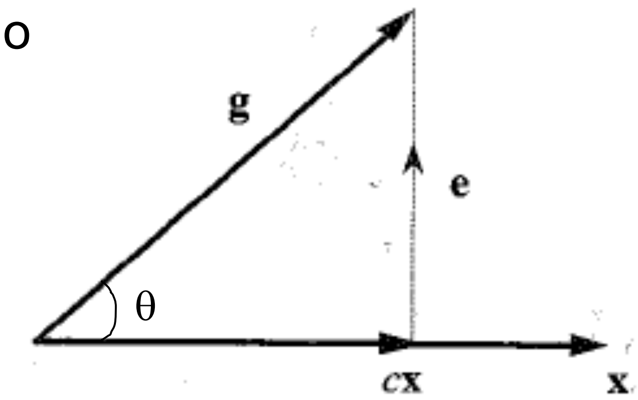


- Low frequencies determines overall shape
- High frequencies determines detail structures

A Vector view of Signal

- To understand why a signal can be represented by linear sum of sinusoidal waveforms, it is useful to consider electrical signals as VECTORS.
- A vector is specified by its magnitude (or length) and its direction.
- Consider two vectors $\mathbf{g}$ and $\mathbf{x}$. If we project $\mathbf{g}$ onto $\mathbf{x}$, we get c$\mathbf{x}$, where c is a scalar (i.e. constant with no direction).
- If we approximate $\mathbf{g}$ with c$\mathbf{x}$, then

  $$\mathbf{g} = c\,\mathbf{x} + \mathbf{e}$$
- $\mathbf{e}$, the error vector, is minimum when it is perpendicular to $\mathbf{x}$.
- c$\mathbf{x}$ is known as the **projection** of $\mathbf{g}$ onto $\mathbf{x}$.



- It can be shown (in the notes below) that:

$$c = \frac{\mathbf{g}\cdot\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}} = \frac{1}{|\mathbf{x}|^2}\,\mathbf{g}\cdot\mathbf{x}$$

Dot product

$$\mathbf{g}\cdot\mathbf{x} = |\mathbf{g}||\mathbf{x}|\cos\theta$$

Orthogonal Set of signals

- If vector **g** is at right angle to vector **x**, then the projection of **g** and **x** is zero. These two vectors (or signals) are known to be **orthogonal**.
- It can easily be shown that two sinusoidal signals of DIFFERENT frequencies are orthogonal to each other.
- The complete set of sinusoidal signals (i.e. of all possible frequency) forms a COMPLETE orthogonal set of signals.
- What this means is that ALL time domain signals can be formed out of projects (or components) onto these these sinusoidal set of signals!
- This is the foundation of **Fourier Series** and **Fourier Transform**, which will be discussed further at the next Lecture.

Three Big Ideas

1. **Time domain** view of a signal is often insufficient. It is often more informative to consider how the signal would appear as a function of frequency, in the **frequency domain**.
2. Any time varying signal can be **decomposed into sinusoidal constituent components** of specific frequencies, phases, and amplitudes, just like the tidal level. This is the main idea of Fourier.
3. Two sinusoidal signals of different frequencies are **orthogonal** to each other, meaning that they have nothing in common, and it is not possible to "produce" one from the other through any linear methods.